Notation: $\mathbb{C}$ are the complex numbers and $\mathbb{N} = \{1, 2, 3, \ldots\}$ are the natural numbers.

You will find below two sets of problems. Neither one is supposed to be simpler. The first is however more labour intensive.

**Solve three out of the four problems below:**

1. For each $n = 0, 1, 2, \ldots$, compute
   \[
   \int_{\Gamma} \frac{\sin^n z}{\ln(1 + z)} \, dz
   \]
   where $\Gamma$ is the positively oriented circle centered around zero of radius $1/2$ (and $\ln$ is the principal branch of the logarithm, $\ln 1 = 0$).

2. Find the complex derivative $F'(z)$ at $z = 0$ for
   \[F(z) = \int_{2/\pi}^{1/\pi} \frac{\sin(e^z/x)}{x} \, dx.\]

3. Show that there cannot be a sequence of analytic functions $f_n$ on $D := \{|z| < 1\}$ that converge uniformly on $D$ to $z^2$ and never vanish in $D$ (i.e. $f_n(z) \neq 0$ for all $z \in D$ and $n \in \mathbb{N}$). (Hint: Use the argument principle.)

4. For what values of $z \in \mathbb{C}$ is the sum of the series
   \[\sum_{n=1}^{\infty} \frac{z^n}{(1 - nz)}\]
   analytic in $z$?
Solve three out of the four problems below:

5. Suppose that \( f \) is entire and \( f'(1/n) = \sin(1/n) \) for all \( n \in \mathbb{N} \). Argue that \( f(z) + \cos(z) \) is a constant function.

6. Let \( f: \mathbb{C} \to \mathbb{C} \) be entire and such that \( |f(z)| \leq \sqrt{|z|} \) whenever \( |z| \geq 1 \). Show that \( |f(z)| \leq 1 \) for all \( z \in \mathbb{C} \). (Hint: Show first that \( f \) is constant.)

7. Suppose that \( f: \mathbb{C} \to \mathbb{C} \) is continuous and that, for all \( z_0, z_1 \in \mathbb{C} \) and any piecewise smooth path \( \Gamma \) from \( z_0 \) to \( z_1 \), we have
   \[
   \int_{\Gamma} f(z) \, dz = e^{z_1^2} - e^{z_0^2}.
   \]
   Prove that \( f \) is analytic. Find the formula for \( f \).

8. Suppose that \( u \) is a non-constant harmonic function on \( \mathbb{C} \). Show that \( \{ u(z) : |z| < 1 \} \) is open. (Hint: Use the maximum principle.)