Complex Analysis PhD Comprehensive Exam
(Jan 2012)

Name:

Pick and circle four out of the five problems below, then solve them. If you rely on a theorem please state it carefully!
Good Luck!

1. Show that the series
\[ f(z) = \sum_{n=1}^{\infty} a_n \sin(nz) \]
defines an entire function if and only if the \( a_n \) converge to 0 super-exponentially, i.e., \( \lim_{n \to \infty} a_n e^{-\alpha n} = 0 \) for all \( \alpha > 0 \). Assume \( a_n > 0 \) for simplicity.

2. Suppose that \( f \) is analytic at \( z = 0 \) and satisfies \( f(z)^2 = z + f(z) \) for all \( z \) in a neighborhood of 0. Determine the radius of convergence of the power series expansion of \( f(z) \) about \( z = 0 \). (Note: Do not compute the series.)

3. Consider the punctured upper half plane \( U \) and the annulus \( A \) as given by
\[ U := \{ z \in \mathbb{C} : \ \text{Im}(z) > 0 \} \setminus \{ i \} \quad \text{and} \quad A := \{ z \in \mathbb{C} : \ 1 < |z| < 2 \} \]
Show that there is an analytic \( f : U \to A \) such that \( f(U) = A \) and then briefly explain why such \( f \) cannot be one-to-one.
Hint: To construct \( f \), you may use a branch of \( F(z) = z^a \) for a suitable \( a > 0 \).

4. Suppose \( D := \{ z \in \mathbb{C} : \ |z| < 1 \} \) and \( f : D \to D \) is analytic. Show that if there are two distinct \( a, b \in D \) with \( f(a) = a \) and \( f(b) = b \) then \( f(z) = z \) for all \( z \in D \).

5. Assuming that \( F : \mathbb{C} \to \mathbb{C} \) is analytic show that
\[ \lim_{R \to \infty} \int_{0}^{R} F(x/R)e^{ix^2} dx = e^{i\pi/4} F(0) \int_{0}^{\infty} e^{-t^2} dt. \]
Hint: Integrate \( F(z/R)e^{iz^2} \) over the triangle with vertices 0, \( R \), and \( R + iR \). Prove that the integrals along the horizontal side and the hypotenuse have the same limit as \( R \to \infty \). Take the limit along the hypotenuse.