PhD Dynamics Comprehensive Examination (Aug 2008)

Name:

Pick and circle five out of the seven following problems, then solve them. Good Luck!

1. Let $L, T > 0$, $x_0 \in \mathbb{R}^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be $L$-Lipshitz. Consider the operator $F$ acting on the space $C([0, T], \mathbb{R}^n)$ of all continuous functions from $[0, T]$ into $\mathbb{R}^n$ according to

$$F(x)(t) := x_0 + \int_0^t \phi(x(s)) \, ds.$$  \hfill (1)

(a) Show that, for some $\lambda \in \mathbb{R}$, $F$ is a contraction on $C([0, T], \mathbb{R}^n)$ with the norm $\|x\| := \sup_{t \in [0, T]} |x(t)| e^{-\lambda t}$.  \hfill (2)

(b) Briefly explain how part a) yields existence, uniqueness and continuous dependence on $x_0$ of the solutions to the initial value problem

$$x(0) = x_0, \quad \dot{x}(t) = \phi(x(t)) \quad \text{for} \ t \in [0, T].$$  \hfill (3)

(c) Conclude from (a) that the solution $x(t)$ of the initial value problem can grow at most exponentially, i.e.

$$\forall x_0 \in \mathbb{R}^n \ \exists C > 0 \ \forall t \in [0, T] \ |x(t)| \leq |x_0|e^{Ct}.$$  \hfill (Note: For more credit show $\exists C > 0 \ \forall x_0 \in \mathbb{R}^n \ \forall t \in [0, T] \ |x(t)| \leq |x_0|e^{Ct}$.)

2. Suppose that $X$ is a manifold, say $\mathbb{R}^n$ or the $n$-dimensional torus $\mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n$. We say that a continuous map $f : X \to X$ is flowable iff there is a continuous flow $(\phi^t)_{t \in \mathbb{R}}$ on $X$ such that $f = \phi^1$.

a) Show that any flowable $f$ is homotopic to the identity.

b) Give an example of $f : \mathbb{T}^1 \to \mathbb{T}^1$ that is not flowable but is homotopic to the identity. Briefly explain why it works.

c) Suppose that $(\phi^t)_{t \in \mathbb{R}}$ is a continuous flow on $\mathbb{R}^n$ and $f := \phi^1 : \mathbb{R}^n \to \mathbb{R}^n$ has all orbits bounded. Carefully show that all orbits of the flow are bounded as well (i.e. $orall x \in \mathbb{R}^n \ \sup_{t \in \mathbb{R}} |\phi^t(x)| < +\infty$).

3. (a) Given an irrational number $\alpha$ and a continuous periodic function $\phi$ with period 1, what can you say about the convergence of the averages $\frac{1}{n} \sum_{k=0}^{n-1} \phi(x+k\alpha)$ as $n \to \infty$? (In what sense do they converge? What is the limit?)

(b) How does your answer change if $\phi$ is only assumed to be Lebesgue integrable?

c) Let $d_1^{(n)} d_2^{(n)} \ldots$ be the decimal expansion of $2^n$. Show that, given any positive digit $D \in \{1, \ldots, 9\}$, there are infinitely many $n \in \mathbb{N}$ such that $d_1^{(n)} = D$. Which digit appears most often?
4. Below, \( f : X \to X \) is a homeomorphism of a compact space.

a) Give an example of \( f \) that has only one \( f \)-invariant ergodic measure. (Indicate why it works.)
b) Is there such an example that is not minimal (i.e. not every orbit is dense)? (Justify without a detailed proof.)
c) For \( f \) being the full shift on two symbols, exhibit (without proof) uncountably many distinct \( f \)-invariant ergodic measures.
d) Show that ergodic \( f \)-invariant measures form a closed set: weak* limit of invariant ergodic measures is invariant and ergodic.

5. Let \( f : X \to X \) and \( g : Y \to Y \) be continuous maps of compact spaces where \( f \) factors to \( g \), i.e., there is a surjective continuous \( h : X \to Y \) with \( h \circ f = g \circ h \).

a) Prove that the topological entropies are related by \( h_{\text{top}}(g) \leq h_{\text{top}}(f) \).
b) Can the inequality be strict if \( h \) is at most \( r \)-to-1 for some \( r \), i.e., \( r : = \sup_{y \in Y} \#h^{-1}(y) < \infty \)? (You do not have to give a detailed proof.)
c) Consider the case when \( f \) is the full shift on \( m \) symbols and \( g : x \mapsto mx \) (mod 1) is an expanding endomorphism of the circle \((m > 1)\). Describe \( h \). What are the topological entropies of \( f \) and \( g \)? What is \( \sup_{y \in Y} \#h^{-1}(y) \)?

6. The cohomological equation \( \phi \circ f - \phi = \psi \) is typically solved for \( \phi \) given \( \psi, f \).

a) State Gottschalk-Hedlund Theorem and show that the solvability condition it supplies is necessary. (This is the easy implication!)
b) For a continuous topologically transitive \( f \), show that any two continuous solutions \( \phi_1 \) and \( \phi_2 \) differ by a constant.
c) Show that continuity of \( \psi \) does not imply continuity of \( \phi \). (Hint: Consider the homoclinic orbits in \( X := \{(x_k)_{k=-\infty}^{\infty} \in \{-1, +1\}^\mathbb{Z} : \forall m,n \in \mathbb{Z} \mid \sum_{k=m}^{n} x_k \mid \leq 3\} \) with the usual shift map \( f : (x_k) \mapsto (x_{k+1}) \) and \( \psi : X \to \mathbb{R} \) given by \( \psi((x_k)) := x_0 \).)

7. A probabilistic Markov chain is a dynamical system \( \sigma : \Omega \to \Omega \) with an invariant Borel probability measure \( \mu \). It is defined by a stochastic matrix \( P = (p_{ij})_{i,j=1}^d \) and a stochastic eigenvector \( p = (p_i)_{i=1}^d \) such that \( pP = p \). (For simplicity, assume that \( P \) is primitive i.e. \( P^n > 0 \) for some \( n > 0 \).)

a) Define \( \Omega, \sigma, \) and \( \mu \) (on the cylinder sets only).
b) Write down the formula for the measure theoretic entropy \( h_\mu(\sigma) \) of \( \sigma \).
c) If \( P \) and \( p \) are allowed to vary with the only proviso that \( \Omega \) remains unchanged, what is the maximal value of \( h_\mu(\sigma) \)? Can you write down the maximizing \( P \) and \( p \)?

If you cannot remember the general formulas at least deal with the case of

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p = (2/3, 1/3), \quad P = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}.
\] (4)
Ad 1. Vanilla.

Ad 2. b) Take an orientation preserving homeomorphism of a circle that has an attracting periodic orbit with rotation number equal to $1/2$.

Ad 3. a) Uniform convergence to the space average, $\int_0^1 \phi(x)dx$.
b) A.e. convergence to the space average.
c) If the first decimal digit of $2^n$ is at the place $m_n$ then $2^n = D10^{m_n} + \cdots = 10^{m_n}(0.D\cdots)$ exactly when $n \log_{10} 2 \in [m_n + \log_{10} D, m_n + \log_{10}(D + 1))$. Taking $\alpha := \log_{10} 2$, one concludes that the first digit of $2^n$ is $D$ iff the fractional part of $n\alpha$ belongs to $[\log_{10} D, \log_{10}(D + 1))$. Since $\alpha$ is irrational, this happens (by part (b)) with density $\log_{10}(D + 1) - \log_{10} D = \log_{10} \frac{D+1}{D}$. Thus 1 is the most common first digit.

Ad 4. a) Any irrational circle rotation.
b) Furstenberg example (see Katok, or Mane).
c) The Bernoulli $(p,q)$-measures where $p + q = 1, p, q > 0$.

Ad 5. a) See Walters.
b) No it cannot. This is a special case of the theorem by Bowen.
c) $h$ is given by the standard coding procedure, see Katok. It is at most 2-to-1. The entropies are $\ln m$.

Ad 6. a) The condition is that $\sup_{n \in \mathbb{N}} |\sum_{k=0}^{n-1} \psi(f^k(x))| < +\infty$. [See Katok page 102]
c) Let $\phi$ be a solution. Consider $x = (+-)^\infty + (+-)^\infty$ so that $f^{2n}x \to y := (+-)^\infty (+-)^\infty$ as $n \to \pm \infty$, that is $x$ is homoclinic to $y$ under $f^2$. If $\phi$ were continuous, then $\phi(f^{2n}(x)) - \phi(f^{-2n}(x)) \to \phi(y) - \phi(y) = 0$. But $\phi(f^{2n}(x)) - \phi(f^{-2n}(x)) = \phi(f^{2n}(x)) - \phi(f^{2n-1}(x)) + \cdots = \psi(f^{2n-1}x) + \cdots + \psi(f^{-2n}x)$ which is equal to 2, a contradiction.

Ad 7. [See Walters]
a) $$\Omega = \{(x_k) : p_{x_kx_{k+1}} > 0\}$$  \hspace{2cm} (5)  
$$\sigma : (x_k) \mapsto (x_{k+1}),$$  \hspace{2cm} (6)  
$$\mu(\{* \cdots x_kx_{k+1} \cdots x_l * * *\}) = p_{x_kp_{x_kx_{k+1}} \cdots p_{x_{l-1}x_l}}$$  \hspace{2cm} (7)  
b) $h_{\mu}(\sigma) = \sum_{ij} -p_{ij}p_{ij} \ln p_{ij}$
c) Let $a_{ij} = 1$ if $p_{ij} > 0$ and $a_{ij} = 0$ otherwise. Then the maximal value of $h_{\mu}(\sigma)$ equals the topological entropy of the subshift associated to the matrix $A := (a_{ij})$, which equals the logarithm $\ln \lambda$ of the spectral radius of $A$. The optimal Markov measure is called the Parry measure and is given by

$$p_i := v_iw_i \quad p_{ij} := \frac{v_ja_{ij}}{\lambda v_i}$$  \hspace{2cm} (8)  
where $\lambda$ is the Perron-Frobenius eigenvalue of $A$ and $Av = \lambda v, wA = \lambda w$ and the dot product of $w$ and $v$ is normalized: $wv = 1$.  
