PhD Dynamics Comprehensive Examination (Aug 2008)

Name:

Pick and circle five out of the seven following problems, then solve them. Good Luck!

1. Let $L, T > 0$, $x_0 \in \mathbb{R}^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be L-Lipshitz. Consider the operator F acting on the space $C([0,T], \mathbb{R}^n)$ of all continuous functions from $[0,T]$ into $\mathbb{R}^n$ according to

$$F(x)(t) := x_0 + \int_0^t \phi(x(s))\, ds. \tag{1}$$

(a) Show that, for some $\lambda \in \mathbb{R}$, F is a contraction on $C([0,T], \mathbb{R}^n)$ with the norm

$$\|x\| := \sup_{t \in [0,T]} |x(t)| e^{-\lambda t}. \tag{2}$$

(b) Briefly explain how part a) yields existence, uniqueness and continuous dependence on x_0 of the solutions to the initial value problem

$$x(0) = x_0, \quad \dot{x}(t) = \phi(x(t)) \;\text{ for } t \in [0,T]. \tag{3}$$

(c) Conclude from (a) that the solution $x(t)$ of the initial value problem can grow at most exponentially, i.e. $\forall_{x_0 \in \mathbb{R}^n}\ \exists_{C>0}\ \forall_{t \in [0,T]}\ |x(t)| \le |x_0| e^{Ct}$.
(*Note:* For more credit show $\exists_{C>0}\ \forall_{x_0 \in \mathbb{R}^n}\ \forall_{t \in [0,T]}\ |x(t)| \le |x_0| e^{Ct}$.)

2. Suppose that X is a manifold, say $\mathbb{R}^n$ or the n-dimensional torus $\mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n$. We say that a continuous map $f : X \to X$ is **flowable** iff there is a continuous flow $(\phi^t)_{t \in \mathbb{R}}$ on X such that $f = \phi^1$.
a) Show that any flowable f is homotopic to the identity.
b) Give an example of $f : \mathbb{T}^1 \to \mathbb{T}^1$ that is not flowable but is homotopic to the identity. Briefly explain why it works.
c) Suppose that $(\phi^t)_{t \in \mathbb{R}}$ is a continuous flow on $\mathbb{R}^n$ and $f := \phi^1 : \mathbb{R}^n \to \mathbb{R}^n$ has all orbits bounded. Carefully show that all orbits of the flow are bounded as well (i.e. $\forall_{x \in \mathbb{R}^n}\ \sup_{t \in \mathbb{R}} |\phi^t(x)| < +\infty$).

3. (a) Given an irrational number α and a continuous periodic function ϕ with period 1, what can you say about the convergence of the averages $\frac{1}{n} \sum_{k=0}^{n-1} \phi(x + k\alpha)$ as $n \to \infty$? (In what sense do they converge? What is the limit?)
(b) How does your answer change if ϕ is only assumed to be Lebesgue integrable?
(c) Let $d_1^{(n)} d_2^{(n)} \ldots$ be the decimal expansion of 2^n. Show that, given any positive digit $D \in \{1, \ldots, 9\}$, there are infinitely many $n \in \mathbb{N}$ such that $d_1^{(n)} = D$. Which digit appears *most often*?

4. Below, $f : X \to X$ is a homeomorphism of a compact space.
a) Give an example of f that has only one f-invariant ergodic measure. (Indicate why it works.)
b) Is there such an example that is not minimal (i.e. not every orbit is dense)? (Justify without a detailed proof.)
c) For f being the full shift on two symbols, exhibit (without proof) uncountably many distinct f-invariant ergodic measures.
d) Show that ergodic f-invariant measures form a closed set: weak* limit of invariant ergodic measures is invariant and ergodic.

5. Let $f : X \to X$ and $g : Y \to Y$ be continuous maps of compact spaces where f factors to g, i.e., there is a surjective continuous $h : X \to Y$ with $h \circ f = g \circ h$.
a) Prove that the topological entropies are related by $h_{top}(g) \leq h_{top}(f)$.
b) Can the inequality be strict if h is at most r-to-1 for some r, i.e., $r := \sup_{y \in Y} \#h^{-1}(y) < \infty$? (You do not have to give a detailed proof.)
c) Consider the case when f is the full shift on m symbols and $g : x \mapsto mx \pmod 1$ is an expanding endomorphism of the circle ($m > 1$). Describe h. What are the topological entropies of f and g? What is $\sup_{y \in Y} \#h^{-1}(y)$?

6. The cohomological equation $\phi \circ f - \phi = \psi$ is typically solved for ϕ given ψ, f.
a) State Gottschalk-Hedlund Theorem and show that the solvability condition it supplies is necessary. (This is the easy implication!)
b) For a continuous topologically transitive f, show that any two continuous solutions ϕ_1 and ϕ_2 differ by a constant.
c) Show that continuity of ψ does not imply continuity of ϕ. (Hint: Consider the homoclinic orbits in $X := \{(x_k)_{k=-\infty}^{\infty} \in \{-1, +1\}^{\mathbb{Z}} : \ \forall_{m,n \in \mathbb{Z}} \ |\sum_{k=m}^{n} x_k| \leq 3\}$ with the usual shift map $f : (x_k) \mapsto (x_{k+1})$ and $\psi : X \to \mathbb{R}$ given by $\psi((x_k)) := x_0$.)

7. A probabilistic Markov chain is a dynamical system $\sigma : \Omega \to \Omega$ with an invariant Borel probability measure μ. It is defined by a stochastic matrix $P = (p_{ij})_{i,j=1}^{d}$ and a stochastic eigenvector $p = (p_i)_{i=1}^{d}$ such that $pP = p$. (For simplicity, assume that P is primitive i.e. $P^n > 0$ for some $n > 0$.)
a) Define Ω, σ, and μ (on the cylinder sets only).
b) Write down the formula for the measure theoretic entropy $h_\mu(\sigma)$ of σ.
c) If P and p are allowed to vary with the only proviso that Ω remains unchanged, what is the maximal value of $h_\mu(\sigma)$? Can you write down the maximizing P and p?
If you cannot remember the general formulas at least deal with the case of

$$p = (2/3, 1/3), \quad P = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

Ad 1. Vanilla.

Ad 2. b) Take an orientation preserving homeomorphism of a circle that has an attracting periodic orbit with rotation number equal to 1/2.

Ad 3. a) Uniform convergence to the space average, $\int_0^1 \phi(x)dx$.

b) A.e. convergence to the space average.

c) If the first decimal digit of 2^n is at the place m_n then $2^n = D10^{m_n} + \cdots = 10^{m_n}(0.D\cdots)$ exactly when $n\log_{10} 2 \in [m_n + \log_{10} D, m_n + \log_{10}(D+1))$. Taking $\alpha := \log_{10} 2$, one concludes that the first digit of 2^n is D iff the fractional part of $n\alpha$ belongs to $[\log_{10} D, \log_{10}(D+1))$. Since α is irrational, this happens (by part (b)) with *density* $\log_{10}(D+1) - \log_{10} D = \log_{10}\frac{D+1}{D}$. Thus 1 is the most common first digit.

Ad 4. a) Any irrational circle rotation.

b) Furstenberg example (see Katok, or Mane).

c) The Bernoulli (p,q)-measures where $p+q=1$, $p,q>0$.

Ad 5. a) See Walters.

b) No it cannot. This is a special case of the theorem by Bowen.

c) h is given by the standard coding procedure, see Katok. It is at most 2-to-1. The entropies are $\ln m$.

Ad 6. a) The condition is that $\sup_{n\in\mathbb{N}} \left|\sum_{k=0}^{n-1} \psi(f^k(x))\right| < +\infty$. [See Katok page 102]

c) Let ϕ be a solution. Consider $x = (+-)^\infty.++(+-)^\infty$ so that $f^{2n}x \to y := (+-)^\infty.(+-)^\infty$ as $n \to \pm\infty$, that is x is homoclinic to y under f^2. If ϕ were continuous, then $\phi(f^{2n}(x)) - \phi(f^{-2n}(x)) \to \phi(y) - \phi(y) = 0$. But $\phi(f^{2n}(x)) - \phi(f^{-2n}(x)) = \phi(f^{2n}(x)) - \phi(f^{2n-1}(x)) + \cdots = \psi(f^{2n-1}x) + \cdots + \psi(f^{-2n}x)$ which is equal to 2, a contradiction.

Ad 7. [See Walters]

a)

$$\Omega = \{(x_k) : \; p_{x_k x_{k+1}} > 0\} \tag{5}$$

$$\sigma : (x_k) \mapsto (x_{k+1}), \tag{6}$$

$$\mu(\{**\cdots x_k x_{k+1} \dots x_l ***\}) = p_{x_k} p_{x_k x_{k+1}} \cdots p_{x_{l-1} x_l} \tag{7}$$

b) $h_\mu(\sigma) = \sum_{ij} -p_i p_{ij} \ln p_{ij}$

c) Let $a_{ij} = 1$ if $p_{ij} > 0$ and $a_{ij} = 0$ otherwise. Then the maximal value of $h_{mu}(\sigma)$ equals the topological entropy of the subshift asociated to the matrix $A := (a_{ij})$, which equals the logarithm $\ln\lambda$ of the spectral radius of A. The optimal Markov measure is called the Parry measure and is given by

$$p_i := v_i w_i \quad p_{ij} := \frac{v_j a_{ij}}{\lambda v_i} \tag{8}$$

where λ is the Perron-Frobenius eigenvalue of A and $Av = \lambda v$, $wA = \lambda w$ and the dot product of w and v is normalized: $wv = 1$.