Ph.D. Comprehensive Exam in Functional Analysis

August 26, 1996

Instructions: Show all work, but be concise. If you recognize that you are asked to prove a familiar theorem, nonetheless, provide a reasonably complete proof. If you find an obvious typo, please correct it and proceed. If you find a serious error or ambiguity, please bring it to the monitor's attention. If you need a definition or help with terminology, such assistance will be provided (in most cases). You have up to four hours and are expected to work the majority of the problems.

1. Suppose that $\mathbf{H}$ and $\mathbf{K}$ are two Hilbert spaces. Prove that one of them is isometrically isomorphic to a closed subspace of the other.

2. Suppose $\{a_n\}_{n\geq 1}$ is a sequence of positive numbers such that $\sum a_n b_n < \infty$ whenever $b_n \geq 0$ and $\sum b_n^2 < \infty$. Show that $\sum a_n^2 < \infty$.

3. Let f denote a bounded linear functional on a subspace M of a Banach space $\mathbf{X}$. We say that f *admits a norm-preserving extension* $g : \mathbf{X} \to \mathbf{C}$ if g is a bounded functional such that

i $||f|| = ||g||$ and

ii $g(x) = f(x)$ for all $x \in M$.

Prove that if $\mathbf{X} = \mathbf{H}$ is a Hilbert space, then any bounded linear functional admits exactly one norm-preserving extension.

4. Let D denote an infinite set. Let $\ell^\infty(D)$ denote the usual Banach space of bounded complex-valued functions defined on D, equipped with the supremum norm. Define $c_0(D)$ to be the set of functions f in $\ell^\infty(D)$ such that for each $\epsilon > 0$, the set $\{x \in D : |f(x)| \geq \epsilon\}$ is finite.

(a) Show that $c_0(D)$ is a closed subspace of $\ell^\infty(D)$.

(b) Prove that $c_0(D)$ is separable iff D is countable.

(c) Prove that $\ell^\infty(D)$ is not separable unless D is finite. Explain how this proves that $c_0(D)$ is not reflexive when D is infinite.

5. Prove that in a normed linear space a complementary space to a one-dimensional subspace is necessarily closed.

6. Let $\mathbf{X}$ denote a normed linear space.

a) Prove that if $K : \mathbf{X} \to \mathbf{X}$ is a compact operator then $K - \lambda I$ has closed range for each complex number $\lambda \neq 0$. Provide complete proofs.

b) Show, by example, that this can be false for $\lambda = 0$.

7. Suppose T is an operator on the Banach space $\mathbf{X}$. Suppose there are constants $\lambda > 0$ and $C > 0$ such that for each $x \in \mathbf{X}$, there exists a number N_x with $||T^n x|| \leq C\,\lambda^n ||x||$, for each $n \geq N_x$. Show that the spectrum of T is contained in a ball of radius λ.

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SKETCHES OF SOLUTIONS.

1. Every Hilbert space admits an orthonormal basis whose cardinality is well-defined. Let B_H and B_K denote these bases, respectively, for H and K. Suppose $\text{card}(B_H) \leq \text{card}(B_K)$. Then there is an injective map $j : B_H \to B_K$. Define the map $T : H \to K$ by setting

$$Tx := T\left(\sum_{r \in B_H} \alpha_r e_r\right) = \sum_{r \in B_H} \alpha_r j(e_r).$$

Evidently, T is an isometry into K, and $T(H)$ is the Hilbert space spanned by the orthonormal subset $T(B_H)$. $\square$

2. The pointwise limit $f(b_1, b_2, \ldots)$ of the functionals $f_k(b_1, b_2, \ldots) = \sum_{n=1}^{k} a_n b_n$ is bounded for each element $\{b_n\}$ of $\ell^2(\mathbf{N})$. By the uniform boundedness principle,

$$\lim ||f_k|| = ||f|| = (\sum a_n^2)^{\frac{1}{2}} < \infty. \ \square$$

3. Let g be the extension which vanishes on $M^\perp$. Then $||g|| = ||f||$. $g(x) =< x, m >$ for some $m \in M$ and all $x \in X$, by the Riesz Theorem. Let h denote a norm-preserving extension of f. Then $h(x) =< x, y >$ for some $y \in X$ and all $x \in X$. We have $||y|| = ||m||$ and we may resolve y as $y = m + m^\perp$. So $||y||^2 = ||m||^2 + ||m^\perp||^2 = ||m||^2$. Thus $m^\perp = 0$ and $y = m$. $\square$.

4. (a) Choose a convergent sequence $\{f_n\}$ in $c_0(D)$ such that $\lim f_n = f$ is in $\ell^\infty(D)$. Let $D_\epsilon = \{x : \ |f(x)| \geq \epsilon\}$. For $n > N$ we may suppose $||f_n - f|| < \epsilon/2$. Then $f_n|_{D_\epsilon}$ has modulus greater than $\epsilon/2$ when $n > N$. But then D_ϵ is finite.

(b) Suppose D is countable. We need to find a countable dense subset. Consider the countable set of all finite rational linear combinations of the characteristic functions of singleton subsets of D.
If D is not countable, the set C of characteristic functions of singleton subsets of D is an uncountable set of functions each of distance ! from any other element of C.

(c) When D is finite with $\text{card}(D) = N$, the spaces $c_0(D)$ and $\ell^\infty(D)$ are simply copies of $\mathbf{C}^N$, which is separable. The set of all countable subsets of D is uncountable. The set of characteristic functions on these countable subsets is uncountable and consists of isolated points. $\square$.

5. Let $M + \text{span}(x) = X$. By HBT or direct construction, there is a continuous functional f with $f|_M = 0$ and $f(x) = ||x||$. Then $\text{Ker}(f) = M$ is closed.

6. One can show that the range of $K - \lambda I$ has finite codimension whenever $\lambda \neq 0$. Subspace of finite codimension in a normed linear space are complemented (by a finite dimen. space). The complement of a one dimensional space is closed by the Hahn-Banach theorem, since it must be the kernel of a bounded functional. Proceed inductively.

7. We know that $\lim ||T^n||^{1/n} = \rho$, the spectral radius. We want to show $\rho \leq \lambda$. Given $\epsilon > 0$, there exists sequences $n_k \to \infty$ and $||x_k|| = 1$ such that $||T^{n_k} x_k||^{1/n_k} \geq \rho - \epsilon$. Then $C^{1/n_k} \lambda \geq \rho - \epsilon$. Which establishes the desired inequality, since ϵ is arbitrary and n_k is arbitrarily large.