FUNCTIONAL ANALYSIS QUALIFYING EXAMINATION

AUGUST 24, 1998

Do as many problems and parts as time allows.

1 a. State the Closed Graph Theorem and the Hahn-Banach Theorem.

b. Use these theorems to prove that if X and Y are Banach spaces and $T : X \to Y$ is a linear map, then T is bounded if and only if for each $y^* \in Y^*$ the composition $y^* \circ T$ is bounded.

2. Let H be a Hilbert space and $A \subset H$ a closed subspace. If $x_0 \in H$ but $x_0 \notin A$, prove that there is bounded linear functional f on H such that $f(x_0) = 1$, $f(a) = 0$ for every $a \in A$ and

$$||f|| = \frac{1}{||x_0 - a_0||} \text{ for a unique } a_0 \in A.$$

3. Let $p \in (1, \infty)$ and suppose that a sequence $\{a_n\}$ of complex numbers has the property that the series

$$\sum_{n=1}^{\infty} a_n b_n$$

is convergent for every $\mathbf{b}(= \{b_n\}) \in l^p$. Show that $\mathbf{a} = \{a_n\}$ is in l^q where $1/p + 1/q = 1$.

4. Show that a non-empty subset S of a normed linear space X is bounded if and only if $f(S)$ is a bounded set of complex numbers for each $f \in X^*$.

5. Consider the equation:

$$u(t) = F(t, M(u))$$

where

$$M(u)(t) = \frac{1}{\epsilon} \int_{t-\frac{\epsilon}{2}}^{t+\frac{\epsilon}{2}} u(s)ds$$

and $F : R^2 \to R$ is a continuous uniformly bounded map that is 1 periodic in t. Show that the above equation has a 1 periodic solution u.

6 a. Let $X = L_2(0,1)$ and $A : D(A) \subset X \to X$ be defined by $D(A) = \{u \in X :$ u is absolutely continuous on $[0,1]$ with $u' \in X$ $\}$ and $A(u) = iu'$. Find the Hilbert space adjoint of A.

b. Let $X = L_2(0,\infty)$ and $B : D(B) \subset X \to X$ be defined by $D(B) = \{u \in X :$ u is absolutely continuous on $[0,\infty)$ with $u' \in X$ and $u(0) = 0$ $\}$ and $B(u) = iu'$. One can show that B is a symmetric graph closed operator with adjoint B^* given by: $D(B^*) = \{u \in X : u$ is absolutely continuous on $[0,\infty)$ with $u' \in X$ $\}$ and $B^*(u) = iu^{'}$ (you need not show this). Show that B doesn't have a self-adjoint extension.

7. Suppose that X is a reflexive Banach space and $F : M \to R$ is weakly sequentially lower semi-continuous where M is a closed bounded convex subset of X. Prove that F attains a minimum on X.

   What can be said if M is not convex?