1. Consider the ODE boundary value problem
\[-\frac{d^2 u}{dx^2} = f(x), \quad 0 < x < 1,\]
\[u(0) + \frac{du}{dx}(0) = 0, \quad u(1) = 0.\]

Recall that the variational, or weak, formulation for this problem is to find \( u \in V \) for which
\[a(u, v) = L(v), \quad \text{for all } v \in V.\]

a. Specify the appropriate vector space \( V \) and derive the functionals \( a(u, v) \) and \( L(v) \) in the variational form.

b. Verify that if \( u \) is a weak solution, and \( u \in C^2[0,1] \), then \( u \) solves the boundary value problem.

c. Explain how to implement the Ritz-Galerkin method for this problem.

2. Consider the parameterized family of time-marching methods
\[u_{j}^{n+1} - u_{j}^{n} = \Delta t \left[ \alpha (u_{j-1}^{n} - 2u_{j}^{n} + u_{j+1}^{n}) + (1 - \alpha) (u_{j-1}^{n+1} - 2u_{j}^{n+1} + u_{j+1}^{n+1}) \right] \]
for the PDE \( u_t = u_{xx} \), \( 0 < x < 1, \ t > 0 \), with homogeneous Dirichlet boundary conditions, where \( u_{j}^{n} \approx u(jh, n\Delta t) \), and \( \alpha \) is real-valued. Determine the ranges of parameters \( \alpha \) for which the time-marching method is unconditionally stable and for which it is always unstable.

For remaining values of \( \alpha \) (where the method is conditionally stable) find conditions on \( h, \Delta t \), and \( \alpha \) which guarantee stability.

3. Consider the difference method
\[\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \frac{u_{i-1,j}^{n} + u_{i+1,j}^{n} + u_{i,j-1}^{n} + u_{i,j+1}^{n} - 4u_{i,j}^{n}}{h^2}\]
for the 2-d heat equation,
\[u_t = u_{xx} + u_{yy}.\]

a. Perform a Von Neumann stability analysis to determine the range of values of \( \Delta t \) and \( h \) for which this method is stable.

b. Provide a general definition of local truncation error for evolution equations, and then compute the local truncation error for this method.
4. Derive the Gauss-Legendre quadrature formula,
\[ \int_{-1}^{1} f(x) \, dx \approx \sum_{k=1}^{n} f(x_k) \, w_k, \]
which is exact for all polynomials of degree \( \leq 2n - 1 \), i.e., explain how to obtain the \( x_k \)'s and \( w_k \)'s, and prove that your formula does indeed have the desired exactness. You may use the Hermite polynomial interpolation formula: If \( f \in C^{2n}[a, b] \) and \( \{x_k\}_{k=1}^{n} \) are distinct points in \([a, b]\), then
\[
f(x) = \sum_{k=1}^{n} f(x_k) h_k(x) + \sum_{k=1}^{n} f'(x_k) \tilde{h}_k(x) + \frac{f^{(2n)}(\xi)}{(2n)!} \left( \prod_{k=1}^{n} (x - x_k)^2 \right),
\]
where for \( k = 1, \ldots, n \),
\[
h_k(x) = [1 - 2(x - x_k)\ell'_k(x_k)][\ell_k(x)]^2, \\
\tilde{h}_k(x) = (x - x_k)[\ell_k(x)]^2, \\
\ell_k(x) = \prod_{i=1,i\neq k}^{n} \frac{(x - x_i)}{(x_k - x_i)}.
\]

5. The leapfrog method for the 1-d linear advection equation
\[ u_t + au_x = 0 \]
is
\[
\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + a \frac{u_{i+1}^{n} - u_{i-1}^{n}}{2h} = 0.
\]
Analyze the stability and accuracy of this method.