Ph.D. Comprehensive Exam: Real Analysis

August 23, 2004

Instructions: Work <u>three</u> of the following four problems. Indicate which problem you are skipping. Work each problem on a separate sheet of paper.

1. Let μ_1 and μ_2 denote measures on the interval $[0,1]$ with σ-algebra $\mathcal{B}$ equal to the Borel subsets of $[0,1]$. Decide which of the following set function rules, if any, determine a measure μ defined on arbitrary Borel sets $A \in \mathcal{B}$. Be sure to justify or explain your answers.

(1) $\mu(A) = \mu_1(A) + \mu_2(A)$
(2) $\mu(A) = \max\{\mu_1(A), \mu_2(A)\}$
(3) $\mu(A) = \mu_1(A) \cdot \mu_2(A)$.
(4) $\mu(A) = \max\{\mu_1(A) - \mu_2(A), 0\}$

2. Evaluate $\displaystyle \lim_{n\to\infty} \int_0^{\frac{n\pi}{2}} \frac{1}{n} e^{-x} \tan\left(\frac{x}{n}\right) dx$. Justify your answer.

3. Does the series $\displaystyle \sum_{n=1}^{\infty} \int_0^{\pi/2} \left(\frac{\pi}{2} - x\right) e^{-nx} \tan x \, dx$ converge to a finite number?

Justify your answer.

4. Suppose μ is a measure on X with σ-algebra $\mathcal{X}$ such that $\mu(X) = 1$. Define the p-norm, as usual:

$$||f||_p = \left\{ \int_X |f|^p \, d\mu \right\}^{\frac{1}{p}}.$$

What can you say about the relation between $||f||_r$ and $||f||_s$ if $1 \le r \le s$? Justify your answer.