Real Analysis
Ph. D. Comprehensive Exam
August 21, 2006

On this exam, all measures are Lebesgue measure.

Do all parts of problem 1 and work two of the last three.

1. Prove or provide a counterexample (with justification).
   (a) If $A \subseteq \mathbb{R}$ is countable then the measure of $A$ is zero.
   (b) If $A \subseteq \mathbb{R}$ has measure zero then $A$ is countable.
   (c) If $f : \mathbb{R} \to \mathbb{R}$ is integrable then $f^2$ is integrable.
   (d) If $f : [0,1] \to \mathbb{R}$ is measurable and $f^2$ is integrable, then $f$ is integrable.
   (e) If $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of measurable functions and $f_n \to f$ a.e., then $f_n \to f$ in measure.

2. Prove that if $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq 6|x - y|$ for all $x, y \in \mathbb{R}$ and $A \subseteq \mathbb{R}$ has measure zero, then $f(A)$ also has measure zero.

3. (a) Give an example of a sequence of $L^1$ functions $g_n$ with $\lim_{n \to \infty} \int_0^1 |g_n| \, d\mu = 0$ but $g_n \not\to 0$ a.e.
   (b) Prove that if $f_n : [0,1] \to \mathbb{R}$ is a sequence of measurable functions with $\int_0^1 |f_n| \, d\mu \leq \frac{1}{n^2}$ for all $n$, then $f_n \to 0$ a.e.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is $L^1$, that $g : \mathbb{R} \to \mathbb{R}$ is continuous and periodic with period 1, and that $\int_0^1 g \, d\mu = 0$. Let $g_n(x) := g(nx)$. Find $\lim_{n \to \infty} \int_{-\infty}^{\infty} f g_n \, d\mu$. (Hint: first consider step functions $f$.)