1. True or false? Justify your answers.

(a) If \( f : \mathbb{R} \to \mathbb{R} \) is Borel measurable, then \( \mu(A) = \int_A f^2 \, d\lambda \) is a Borel measure.

(b) If \( f : \mathbb{R} \to \mathbb{R} \) is non-negative and Borel measurable, then \( \mu(A) = (\int_A f \, d\lambda)^2 \) is a Borel measure.

(c) If \( f_n \to f \) in \( L_2([0, 1]) \), then this convergence also holds in \( L_1([0, 1]) \).

(d) If \( f_n \to f \) in \( L_2([0, 1]) \), then this convergence also holds in \( L_\infty([0, 1]) \).

(e) There exists a Borel charge \( \mu \) on \( \mathbb{R} \) with \( \mu((a, b]) = \sin b - \sin a \) for all \( a, b \in \mathbb{R}, \ a < b \).

2. Find (with justification)

\[
\lim_{n \to \infty} \int_0^1 \cos \frac{1}{nx} \, dx.
\]

3. Let \( f, g : [0, 1] \to [0, \infty) \) be Borel measurable functions with \( f(x)g(x) \geq 1 \) for (Lebesgue) almost every \( x \in [0, 1] \). Show that

\[
\int_{[0,1]} f \, d\lambda \cdot \int_{[0,1]} g \, d\lambda \geq 1.
\]

(Hint: Cauchy-Schwarz inequality.)

4. Let \( f, g : \mathbb{R} \to [0, \infty) \) be Borel measurable functions, and let \( \mu(A) = \int_A f \, d\lambda, \ \nu(B) = \int_B g \, d\lambda \) for Borel sets \( A, B \subseteq \mathbb{R} \).

(a) Show that \( \mu \) and \( \nu \) are \( \sigma \)-finite Borel measures absolutely continuous with respect to Lebesgue measure.

(b) Let \( \pi \) be the product measure of \( \mu \) and \( \nu \) on \( \mathbb{R}^2 \). Find the Radon-Nikodym derivative of \( \pi \) with respect to two-dimensional Lebesgue measure \( \lambda^2 \). Justify your answer.