REAL ANALYSIS QUALIFYING EXAMINATION QUESTIONS

1. True or False? If True, prove; if False, give a counterexample.
   a. If $A$ is a subset of $R$ and $m^*(A) = 0$, then for any $a, b \in A$, $a < b$, there is an $x \in R - A$ such that $a < x < b$. (Here $m^*$ is Lebesgue outer measure.)
   b. If $A$ is a subset of $[0, 1]$ and $m^*(A) = 1$, then $A$ contains an interval of positive length. (Here $m^*$ is Lebesgue outer measure.)
   c. If $E \subset [0, 1]$ satisfies $m^*(E) + m^*([0, 1] - E) = 1$ then $E$ is measurable. (Here $m^*$ is Lebesgue outer measure.)
   d. Let $(X, M, \mu)$ be a measure space, and let $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ be a sequence in $M$. Then $\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \mu(A_n)$.

Do any three of the following four questions:

2. Let $(X, \mu)$ be a measure space and let $f$ be a measurable function on $X$.
   (a) Assume $f$ is bounded and $f \in L^1(X)$. Prove that $f \in L^2(X)$.
   (b) Assume $\mu(X) < \infty$, and $f \in L^2(X)$. Prove that $f \in L^1(X)$.

3. Let $f : R \to R$ be a positive function. Define $X = \{(x, y) \in R^2 | 0 \leq y \leq f(x)\}$
   Assume that $X$ is Lebesgue measurable and its measure is finite. Show that the function $f$ is integrable, and the measure of $X$ is equal to $\int_R f(x)dx$.

4. Suppose that $E$ is a Lebesgue measurable subset of $R$ and that the Lebesgue measure of $E$ is finite. Suppose also that for each $n \in \mathbb{N}$, $f_n : E \to R$ is a bounded, measurable, non-negative function on $E$ and that $\lim_{n \to \infty} f_n(x)$ exits in $R$. Let $f(x) = \lim_{n \to \infty} f_n(x)$. Show by counterexample that without further hypotheses, the following assertion is false:
   $\int_E f(x)dx = \lim_{n \to \infty} \int_{E} f_n(x)dx$
   What additional hypothesis are required to make the above assertion true (but still non trivial)?

5. If $E$ is a measurable set in $R^d$ and $x \in R^d$, we say that $x$ is a point of Lebesgue density of $E$ if
   $\lim_{r \to 0^+} \frac{m(B_r(x) \cap E)}{m(B_r(x))} = 1$
   where $B_r(x)$ is the ball in $R^d$ centered at $x$ with radius $r$ and $m$ is Lebesgue measure.
   Suppose that 0 is a point of Lebesgue density of $A$ and of $B$, both measurable subsets of $R^d$. Show that 0 is also point of Lebesgue density of $A \cap B$.  