Real Analysis Ph. D. Comprehensive Exam

August 2011

Do all parts of problem 1, and work 3 of the other 4 problems.

If not explicitly specified, the measure space is $\mathbb{R}$
with the Borel σ-algebra $\mathbf{B}$ and Lebesgue measure λ.

1. True or false? Justify your answers.

(a) If (f_n) is a sequence of measurable non-negative functions on a measure space $(X, \mathbf{X}, \mu)$, then $\displaystyle\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$.

(b) If μ_1 and μ_2 are σ-finite measures with $\mu_1 \ll \mu_2$, then there exists a μ_2-integrable function f with $\mu_1(A) = \int_A f \, d\mu_2$ for all measurable sets A.

(c) For every $f \in L_2$ there exists a sequence of functions $\phi_n \in L_2$, vanishing outside $[-n, n]$, such that $\phi_n \to f$ in L_2.

(d) For every $f \in L_\infty$ there exists a sequence of functions $\phi_n \in L_\infty$, vanishing outside $[-n, n]$, such that $\phi_n \to f$ in L_∞.

2. Let $(X, \mathbf{X}, \mu)$ be a measure space, and let $(E_k)_{k=1}^{\infty}$ be a sequence of measurable sets such that $\mu(E_k) \geq 2011$ for all $k \in \mathbb{N}$. Let E be the set of points in X which belong to E_k for infinitely many indices k.

(a) Show that E is measurable.

(b) Under the additional assumption that $\mu(X) < \infty$, show that $\mu(E) \geq 2011$.

(c) Give an example to show that the additional assumption in (b) is necessary.

3. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of Borel-measurable functions converging to 0 uniformly, and satisfying $\int_{\mathbb{R}} |f_n| \, d\lambda \leq 1$ for all n.

(a) Show that $f_n \to 0$ in L_p for all $p > 1$.

(b) Give an example to show that the assumptions do not imply $f_n \to 0$ in L_1.

4. For $\alpha \geq 0$ let $F(\alpha) = \displaystyle\int_0^1 \frac{x^\alpha - 1}{\ln x} \, dx$.

(a) Show that F is differentiable with $F'(\alpha) = \dfrac{1}{\alpha + 1}$.

(b) Use this result to calculate $\displaystyle\int_0^1 \frac{x - 1}{\ln x} \, dx$.

5. Let f be a non-negative measurable function on a σ-finite measure space $(X, \mathbf{X}, \mu)$. Show that $\displaystyle\int_X f \, d\mu = \int_0^\infty \mu(F_t) \, dt$, with $F_t = \{x \in X : f(x) > t\}$.