Real Analysis Ph. D. Comprehensive Exam
August 2011

Do all parts of problem 1, and work 3 of the other 4 problems.

If not explicitly specified, the measure space is \( \mathbb{R} \) with the Borel \( \sigma \)-algebra \( \mathcal{B} \) and Lebesgue measure \( \lambda \).

1. True or false? Justify your answers.

(a) If \( (f_n) \) is a sequence of measurable non-negative functions on a measure space \((X, X, \mu)\), then \( \int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu \).

(b) If \( \mu_1 \) and \( \mu_2 \) are \( \sigma \)-finite measures with \( \mu_1 \ll \mu_2 \), then there exists a \( \mu_2 \)-integrable function \( f \) with \( \mu_1(A) = \int_A f \, d\mu_2 \) for all measurable sets \( A \).

(c) For every \( f \in L^2 \) there exists a sequence of functions \( \phi_n \in L^2 \), vanishing outside \([-n,n]\), such that \( \phi_n \to f \) in \( L^2 \).

(d) For every \( f \in L^\infty \) there exists a sequence of functions \( \phi_n \in L^\infty \), vanishing outside \([-n,n]\), such that \( \phi_n \to f \) in \( L^\infty \).

2. Let \((X, X, \mu)\) be a measure space, and let \( (E_k)_{k=1}^{\infty} \) be a sequence of measurable sets such that \( \mu(E_k) \geq 2011 \) for all \( k \in \mathbb{N} \). Let \( E \) be the set of points in \( X \) which belong to \( E_k \) for infinitely many indices \( k \).

(a) Show that \( E \) is measurable.

(b) Under the additional assumption that \( \mu(X) < \infty \), show that \( \mu(E) \geq 2011 \).

(c) Give an example to show that the additional assumption in (b) is necessary.

3. Let \( f_n : \mathbb{R} \to \mathbb{R} \) be a sequence of Borel-measurable functions converging to 0 uniformly, and satisfying \( \int_{\mathbb{R}} |f_n| \, d\lambda \leq 1 \) for all \( n \).

(a) Show that \( f_n \to 0 \) in \( L^p \) for all \( p > 1 \).

(b) Give an example to show that the assumptions do not imply \( f_n \to 0 \) in \( L^1 \).

4. For \( \alpha \geq 0 \) let \( F'(\alpha) = \int_0^1 \frac{x^{\alpha} - 1}{\ln x} \, dx \).

(a) Show that \( F \) is differentiable with \( F'(\alpha) = \frac{1}{\alpha + 1} \).

(b) Use this result to calculate \( \int_0^1 \frac{x - 1}{\ln x} \, dx \).

5. Let \( f \) be a non-negative measurable function on a \( \sigma \)-finite measure space \((X, X, \mu)\).

Show that \( \int_X f \, d\mu = \int_0^\infty \mu(F_t) \, dt \), with \( F_t = \{ x \in X : f(x) > t \} \).