Ph.D. Comprehensive Examination
Topology
August 26, 2002

1. Suppose that (X, d) is a metric space, that C_1 is a closed subset of X, that C_2 is a compact subset of X, and that $C_1 \cap C_2 = \emptyset$.

   Let $r = \inf\{d(x_1, x_2) : x_1 \in C_1,\ x_2 \in C_2\}$.

   a) Prove that $r > 0$.

   b) Is it necessarily the case that $d(x_1, x_2) = r$ for some $x_1 \in C_1,\ x_2 \in C_2$?

2. Let C be a closed subset of $\mathbb{R}$ (the reals) with empty interior and let $S = \{x \in \mathbb{R} : x + n \notin C \text{ for all integers } n\}$. Prove that S is dense in $\mathbb{R}$. (Hint: $\mathbb{R}$ is a Baire space.)

3. Let M be a connected 3-manifold (that is, M is a connected topological space and every point of M has a neighborhood homeomorphic with $\mathbb{R}^3$, Euclidean 3-space.) Prove that M is path connected.

4. Let $\mathbb{D}^2 = \{(x, y) : x^2 + y^2 \leq 1\}$ and, for $0 \leq r \leq 1$, let $S_r = \{(x, y) : x^2 + y^2 = r^2\}$. Suppose that $f : \mathbb{D}^2 \to \mathbb{D}^2$ is continuous and that, for some nonzero interger n, $f(\cos\theta, \sin\theta) = (\cos n\theta, \sin n\theta)$ for all θ.

   a) Prove that f is a surjection.

   b) If, in addition, $f^{-1}(\{0, 0\}) = \{(0, 0)\}$ and $|n| \geq 2$ prove that $f|_{S_r}$ is not one-to-one for any r, $0 < r \leq 1$.

5. Let K and L be simplicial complexes with $|K|$ homeomorphic with the 2-dimensional torus and $|L|$ homeomorphic with the 2-dimensional sphere. Suppose that K and L share a 2-simplex (that is, $K \cap L$ consists of a 2-simplex and its faces). Use exact sequences to compute the homology groups $H_p(K \cup L)$ and $H_p(K \cup L, K \cap L)$, all $p \geq 0$.