PhD Comprehensive Exam
Topology
August 25, 2006 (08:00-12:00)

Problems 1-3 are required.

Problem 1. Suppose that X is a compact Hausdorff space. Prove that for any topological space Y, the projection $X \times Y \xrightarrow{p} Y$ is a closed map.

Problem 2. Suppose that A is a compact subspace of the topological space X and B is a compact subspace of the topological space Y, and suppose that W is an open subset of the product space $X \times Y$ which contains the set $A \times B$. Prove that there exist open sets U and V, $U \subset X$ and $V \subset Y$, such that

$$A \times B \subset U \times V \subset W \subset X \times Y$$

Problem 3. Suppose that X is a compact, locally connected, Hausdorff space and C is a non-empty, closed, connected subset of X. Prove that for any open subset U of X containing C there exists a connected, open set V such that $C \subset V \subset \bar{V} \subset U$.

Choose two of the following four problems.

Problem 4. Find examples to verify the following statements.

(a) There is a continuous injection (one-to-one) $X \xrightarrow{f} Y$ for which the induced homomorphism of fundamental groups $\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0)$ is not a monomorphism.

(b) There is a continuous surjection (onto) $X \xrightarrow{g} Y$ for which the induced homomorphism of fundamental groups $\pi_1(X, x_0) \xrightarrow{g_*} \pi_1(Y, y_0)$ is not an epimorphism.

Problem 5. Find a covering space $E \xrightarrow{p} T$ of the torus $T = S^1 \times S^1$ for which the subgroup $p_* (\pi_1(E))$ has index 3 in the group[1] $\pi_1(T)$, with appropriately chosen base points.

[1]The index of a subgroup is the number of its left (or right) cosets.

Problem 6. Let K denote the simplicial complex shown schematically in Fig. 1: K consists of *four* vertices (0-simplices), *five* 1-simplices, and *one* 2-simplex (cross-hatched in the figure). The subcomplex L is just the 1-skeleton of K.

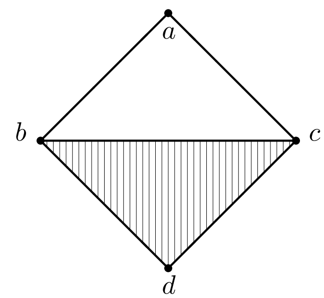

Figure 1: (K, L)

(a) Find the simplicial homology groups $H_p(K)$, $p = 0, 1, 2, \ldots$ (with integer coefficients).

(b) Find the relative simplicial homology groups $H_p(K, L)$, $p = 1, 2, \ldots$ (with integer coefficients).

Problem 7. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map and suppose that for some constant $r > 0$ the following holds[2]

$$\|x\| = r \quad \Longrightarrow \quad \|f(x)\| \leq r$$

Prove that f has a fixed point x_0, with $\|x_0\| \leq r$.

[2]the norm is the usual Euclidean norm:

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$