In the following, \( \mathbb{R} \) is the set of real numbers. Every answer to a question in this exam needs a proof.

1. Let \( \tau \) be the collection of subsets \( U \) of \( \mathbb{R} \) such that \( U \supset (0, 1) \), together with the empty set.
   
   \( \textbf{(a)}: \) Show that \( \tau \) is a topology on \( \mathbb{R} \).
   
   \( \textbf{(b)}: \) Find the closure of the interval \((0, 1)\) in \((\mathbb{R}, \tau)\).
   
   \( \textbf{(c)}: \) Find the interior of the interval \((0, 1)\) in \((\mathbb{R}, \tau)\).
   
   \( \textbf{(d)}: \) Let \((\mathbb{R}, u)\) denote \( \mathbb{R} \) with its usual topology. Is the function \( f : (\mathbb{R}, u) \to (\mathbb{R}, \tau) \) defined by \( f(x) = x \) continuous? Is the function \( g : (\mathbb{R}, \tau) \to (\mathbb{R}, u) \), defined by \( g(x) = x \) continuous?

2. Let \( f : X \to Y \) be a continuous map of Hausdorff spaces. Let \( B_1 \supset B_2 \supset \ldots \supset B_n \supset \ldots \) be a decreasing sequence of compact subsets of \( X \). Prove that

\[
  f\left(\bigcap_{i=1}^{\infty} B_i\right) = \bigcap_{i=1}^{\infty} f(B_i).
\]

3. (a) Show that any continuous \( f : S^2 \to S^1 \times S^1 \) must be null-homotopic (i.e., homotopic to a constant map).

   (b) Show that there is a continuous map \( g : S^1 \times S^1 \to S^2 \) that is not null-homotopic.

4. View the Klein bottle \( K \) as a union of two Mobius bands \( M_1 \) and \( M_2 \) identified along their boundaries. Compute \( H_*(K) \).