Proof:

The key to proving this theorem is the last exercise in the exercise set above.

Let

$$\vec{e}_j = (0, \ldots, 0, \underbrace{1}_{i^{\text{th}} \text{ place}}, 0, \ldots, 0)$$

and let

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{kj} \end{bmatrix} = T(\vec{e}_j).$$

That is, the jth column of the matrix A is $T(\vec{e}_j)$ written as a column vector.

Now if $\vec{x}$ is any vector in R^n we can write

$$\vec{x} = \sum_{j=1}^{n} x_j \vec{e}_j$$

and since T is linear

$$\begin{aligned} \vec{y} &= T(\vec{x}) \\ &= T\left(\sum_{j=1}^{n} x_j \vec{e}_j\right) \\ &= \sum_{j=1}^{n} x_j T(\vec{e}_j) \\ &= \sum_{j=1}^{n} x_j \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{kj} \end{bmatrix} \end{aligned}$$

and, in particular,

$$y_i = \sum_{j=1}^{n} a_{ij} x_j$$

So that

$$\vec{y} = A\vec{x}$$

as we wanted to show. ■