Theorem 4.11 *Suppose that x_1 and x_2 are independent random numbers and that the pdf for x_1 is ϕ_{σ_1} and the pdf for x_2 is ϕ_{σ_2}. Then the pdf for $y = x_1 + x_2$ is ϕ_σ where $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$.*

Proof:

We give a proof for the case when $\sigma_1 = \sigma_2 = 1$ and, hence, $\sigma = \sqrt{2}$. The proof involves straightforward algebraic manipulations. The general proof involves similar but more tedious algebraic manipulations. Let $\phi(x)$ denote $\phi_1(x)$.

Let $\psi(y)$ denote the pdf for y. We want to show that

$$\psi(y) = \phi_{\sqrt{2}}(y) = \frac{1}{2\sqrt{\pi}} e^{-\frac{y^2}{4}}.$$

Now,

$$\begin{aligned}
\psi(y) &= (\phi * \phi)(y) \\
&= \int_{-\infty}^{+\infty} \left(\frac{1}{\sqrt{2\pi}}\right) e^{-\frac{s^2}{2}} \left(\frac{1}{\sqrt{2\pi}}\right) e^{-\frac{(y-s)^2}{2}}\ ds \\
&= \left(\frac{1}{2\pi}\right) \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} e^{-\frac{(y-s)^2}{2}}\ ds \\
&= \left(\frac{1}{2\pi}\right) \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} e^{-\frac{(y-s)^2}{2}} e^{\frac{y^2}{4}} e^{-\frac{y^2}{4}}\ ds \\
&= \left(\frac{1}{2\pi}\right) e^{-\frac{y^2}{4}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2}} e^{-\frac{(y-s)^2}{2}} e^{\frac{y^2}{4}}\ ds \\
&= \left(\frac{1}{2\pi}\right) e^{-\frac{y^2}{4}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{2} - \frac{(y-s)^2}{2} + \frac{y^2}{4}}\ ds \\
&= \left(\frac{1}{2\pi}\right) e^{-\frac{y^2}{4}} \int_{-\infty}^{+\infty} e^{-\left(s-\frac{y}{2}\right)^2}\ ds.
\end{aligned}$$

Now

$$\int_{-\infty}^{+\infty} e^{-\left(s-\frac{y}{2}\right)^2}\ ds = \int_{-\infty}^{+\infty} e^{-\frac{\left(s\sqrt{2}-\frac{y}{\sqrt{2}}\right)^2}{2}}\ ds$$

and by the substitution

$$v = \left(s\sqrt{2} - \frac{y}{\sqrt{2}}\right)$$

so that

$$ds = \left(\frac{1}{\sqrt{2}}\right)\ dv$$

we see that

$$\int_{-\infty}^{+\infty} e^{-\frac{\left(s\sqrt{2}-\frac{y}{\sqrt{2}}\right)^2}{2}}\, ds = \sqrt{\pi},$$

so

$$\psi(y) = \frac{1}{2\sqrt{\pi}} e^{-\frac{y^2}{4}}$$

which was what we wanted to show. ■