Theorem 4.2

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x^2}{2}\right)}\, dx = 1.$$

Proof:

We will evaluate the integral

$$\int_{R^2} \frac{1}{2\pi} e^{-\left(\frac{x^2+y^2}{2}\right)}\, dA$$

two ways.

(a) First, we evaluate this integral as it is written in Cartesian coordinates.

$$\begin{aligned}
&(4.1) && \int_{R^2} \frac{1}{2\pi} e^{-\left(\frac{x^2+y^2}{2}\right)}\, dA \\
&(4.2) && = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-\left(\frac{x^2+y^2}{2}\right)}\, dx\ dy \\
&(4.3) && = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\left(\frac{x^2}{2}\right)} e^{-\left(\frac{y^2}{2}\right)}\, dx\ dy \\
&(4.4) && = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[e^{-\left(\frac{y^2}{2}\right)} \int_{-\infty}^{+\infty} e^{-\left(\frac{x^2}{2}\right)}\, dx \right] dy \\
&(4.5) && = \frac{1}{2\pi} \left[\int_{-\infty}^{+\infty} e^{-\left(\frac{x^2}{2}\right)}\, dy \right] \left[\int_{-\infty}^{+\infty} e^{-\left(\frac{y^2}{2}\right)}\, dx \right] \\
&(4.6) && = \frac{1}{2\pi} \left[\int_{-\infty}^{+\infty} e^{-\left(\frac{x^2}{2}\right)}\, dx \right] \left[\int_{-\infty}^{+\infty} e^{-\left(\frac{x^2}{2}\right)}\, dx \right] \\
&(4.7) && = \frac{1}{2\pi} \left[\int_{-\infty}^{+\infty} e^{-\left(\frac{x^2}{2}\right)}\, dx \right]^2
\end{aligned}$$

where line (4.4) follows from line (4.3) from the observation that y is a constant during the inner integration and line (4.6) follows from line (4.5) by replacing the variable y by the variable x.

(b) Second, we evaluate the same integral in cylindrical coordinates.

$$
\begin{aligned}
&\int_{R^2} \frac{1}{2\pi} e^{-\left(\frac{x^2+y^2}{2}\right)}\ dA \\
&= \int_{R^2} \frac{1}{2\pi} e^{-\left(\frac{r^2}{2}\right)}\ dA \\
&= \int_0^{2\pi} \int_0^{+\infty} \frac{1}{2\pi} e^{-\left(\frac{r^2}{2}\right)} r\ dr\ d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{+\infty} e^{-\left(\frac{r^2}{2}\right)} r\ dr\ d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left[-e^{-\frac{r^2}{2}}\right]_0^{+\infty}\ d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} 1\ d\theta \\
&= 1
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{1}{2\pi} \left[\int_{-\infty}^{+\infty} e^{-\left(\frac{x^2}{2}\right)}\ dx\right]^2 &= 1 \\
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\left(\frac{x^2}{2}\right)}\ dx &= 1. \ \blacksquare
\end{aligned}
$$