Theorem 4.12 *Suppose that x_1 and x_2 are independent random numbers. Then*

$$\text{var}(x_1 + x_2) = \text{var}(x_1) + \text{var}(x_2).$$

Proof:

Let $\psi_1(x_1)$ be the pdf for x_1 and $\psi_2(x_2)$ be the pdf for x_2. To simplify the proof we will assume that x_1 and x_2 have mean zero. This implies that $x_1 + x_2$ has mean zero as well. The proof is a little more difficult without the assumption that x_1 and x_2 have mean zero.

Thus,

$$\begin{aligned}
&\text{var}(x_1 + x_2) \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x_1 + x_2)^2 \psi_1(x_1)\psi_2(x_2)\ dx_1\ dx_2 \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x_1^2 + 2x_1x_2 + x_2^2)\psi_1(x_1)\psi_2(x_2)\ dx_1\ dx_2 \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1^2\psi_1(x_1)\psi_2(x_2)\ dx_1\ dx_2 + \\
&\quad \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} 2x_1x_2\psi_1(x_1)\psi_2(x_2)\ dx_1\ dx_2 + \\
&\quad \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_2^2\psi_1(x_1)\psi_2(x_2)\ dx_1\ dx_2.
\end{aligned}$$

But

$$\begin{aligned}
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1^2\psi_1(x_1)\psi_2(x_2)\ dx_1\ dx_2 &= \int_{-\infty}^{+\infty} \psi_2(x_2) \left[\int_{-\infty}^{+\infty} x_1^2\psi_1(x_1)\ dx_1\right]\ dx_2 \\
&= \int_{-\infty}^{+\infty} \psi_2(x_2)\text{var}(x_1)\ dx_2 \\
&= \text{var}(x_1) \int_{-\infty}^{+\infty} \psi_2(x_2)\ dx_2 \\
&= \text{var}(x_1)
\end{aligned}$$

and

$$\begin{aligned}
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_2^2\psi_1(x_1)\psi_2(x_2)\ dx_1\ dx_2 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_2^2\psi_1(x_1)\psi_2(x_2)\ dx_2\ dx_1 \\
&= \int_{-\infty}^{+\infty} \psi_1(x_1) \left[\int_{-\infty}^{+\infty} x_2^2\psi_2(x_2)\ dx_2\right]\ dx_1 \\
&= \int_{-\infty}^{+\infty} \psi_1(x_1)\text{var}(x_2)\ dx_1 \\
&= \text{var}(x_2) \int_{-\infty}^{+\infty} \psi_1(x_1)\ dx_1 \\
&= \text{var}(x_2).
\end{aligned}$$

But,

$$\begin{aligned}
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 \psi_1(x_1)\psi_2(x_2)\ dx_1\ dx_2 &= \int_{-\infty}^{+\infty} x_2\psi_2(x_2)\left[\int_{-\infty}^{+\infty} x_1\psi_1(x_1)\ dx_1\right]\ dx_2 \\
&= \int_{-\infty}^{+\infty} x_2\psi_2(x_2) 0\ dx_2 \\
&= 0.
\end{aligned}$$

Putting these all together we see that

$$\begin{aligned}
\mathrm{var}(x_1 + x_2) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x_1^2 + 2x_1x_2 + x_2^2)\psi_1(x_1)\psi_2(x_2)\ dx_1\ dx_2 \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1^2\psi_1(x_1)\psi_2(x_2)\ dx_1\ dx_2 + \\
&\quad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} 2x_1x_2\psi_1(x_1)\psi_2(x_2)\ dx_1\ dx_2 + \\
&\quad \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_2^2\psi_1(x_1)\psi_2(x_2)\ dx_1\ dx_2 \\
&= \mathrm{var}(x_1) + 2\cdot 0 + \mathrm{var}(x_2) \\
&= \mathrm{var}(x_1) + \mathrm{var}(x_2)\ \blacksquare
\end{aligned}$$