Singular boundary value problems via the Conley index

Tomáš Gedeon*
Department of Mathematical Sciences
Montana State University
Bozeman, MT 59717-0240, U.S.A.
gedeon@math.montana.edu

Konstantin Mischaikow†
Center for Dynamical Systems and Nonlinear Studies
School of Mathematics
Georgia Institute of Technology, Atlanta, GA 30332, U.S.A.
mischaik@math.gatech.edu

March 16, 2005

Abstract

We use Conley index theory to solve the singular boundary value problem $\epsilon^2 D u_{xx} + f(u, \epsilon u_x, x) = 0$ on an interval $[-1, 1]$, where $u \in \mathbf{R}^n$ and D is a diagonal matrix, with separated boundary conditions. Since we use topological methods the assumptions we need are weaker then the standard set of assumptions. The Conley index theory is used here not for detection of an invariant set, but for tracking certain cohomological information, which guarantees existence of a solution to the boundary value problem.

Key words: Singular boundary value problems, Conley index.

*Research was supported in part by NSF EIA-BITS grant 426411.
†Research was supported in part by NSF grant DMS-0107395.

1 Introduction

We study a singular boundary value problem for the system of equations

$$\epsilon^2 D u_{xx} + f(u, \epsilon u_x, x) = 0 \tag{1}$$

on the interval $\Lambda := [-1, 1]$, $u \in \mathbf{R}^n$ and D a diagonal matrix, with boundary conditions

$$\begin{aligned} (u_x(-1), u(-1)) &\in A \\ (u_x(1), u(1)) &\in B, \end{aligned} \tag{2}$$

where A and B are manifolds in $\mathbf{R}^{2n}$. These boundary conditions generalize linear boundary conditions of the type $a_1 u_x(-1) + a_2 u(-1) = a_3$ and $b_1 u_x(1) + b_2 u(1) = b_3$, where $a_i \in \mathbf{R}^k$, $b_i \in \mathbf{R}^l$ with $l + k = 2n$.

We set $\lambda := x$ and rewrite the system as

$$\begin{aligned} \epsilon u_x &= v \\ \epsilon v_x &= -f(u, \epsilon u_x, \lambda) \\ \lambda_x &= 1. \end{aligned}$$

Setting $t := \epsilon x$ we get

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -f(u, v, \lambda) \\ \dot{\lambda} &= \epsilon. \end{aligned} \tag{3}$$

The system of equations (3) is a special case of fast-slow systems on $\mathbf{R}^{2n} \times \mathbf{R}$ given by

$$\begin{aligned} \dot{u} &= h(u, \lambda) \\ \dot{\lambda} &= \varepsilon g(u, \lambda) \end{aligned}$$

where $f(u, \lambda) : \mathbf{R}^{2n} \times \mathbf{R} \to \mathbf{R}^{2n}$ and $g(u, \lambda) : \mathbf{R}^{2n} \times \mathbf{R} \to \mathbf{R}$ are C^1 functions and $\varepsilon \geq 0$. The solutions to this equation generate a flow

$$\varphi^\varepsilon : \mathbf{R} \times \mathbf{R}^{2n} \times \mathbf{R} \to \mathbf{R}^{2n} \times \mathbf{R}.$$

In the special case $\varepsilon = 0$, (3) has a simpler form since λ is a constant. We can view λ as a parameter for flows on $\mathbf{R}^{2n}$, and for each λ we define a flow ψ^λ on $\mathbf{R}^{2n}$ by

$$(\psi^\lambda(u), \lambda) = \varphi^0(t, u, \lambda). \tag{4}$$

For a fixed range of values of $\lambda \in \Lambda$ equation (4) defines a *parameterized flow*

$$\psi^\Lambda : \mathbf{R} \times \mathbf{R}^{2n} \times \Lambda \to \mathbf{R}^{2n} \times \Lambda$$

given by $\psi^\Lambda(t, u, \lambda) := (\psi^\lambda(t, u), \lambda)$.

Consider for the moment an arbitrary flow γ defined on a locally compact metric space X. A compact set $N \subset X$ is an *isolating neighborhood* if

$$\text{Inv}(N, \gamma) := \{x \in X \mid \gamma(\mathbf{R}, x) \subset N\} \subset \text{int}N,$$

where Inv denotes the maximal invariant set in N and int$N$ is the interior of N. If $S = \text{Inv}(N, \gamma)$ for some isolating neighborhood N, then S is referred to as an *isolated invariant set*. The Conley index is an index of isolating neighborhoods with the property that if $\text{Inv}(N, \gamma) = \text{Inv}(N', \gamma)$ then the Conley index of N equals the Conley index of N'. In this way one may, also, view the Conley index as an index of isolated invariant sets. Given an isolating neighborhood, its Conley index can be used to describe the associated isolated invariant set. The goal of this paper is to show that it can also be used to solve singular boundary value problems. We now describe the intuition behind the construction of the solutions of (1) and (2) for small ϵ. These solutions will be close to singular "solutions" of (3), constructed by the concatenation of branches of invariant sets in the parameterized flow ψ^Λ and connecting orbits between such branches in the flow $\varphi^0(t, u, \lambda)$ for a particular value of λ. Each inner layer of the solution is close to such a connecting orbit, while the boundary layers approximate the stable manifold of an invariant set of the flow $\psi^{(-1)}$ and the unstable manifold of an invariant set of ψ^1, see Figure 1.

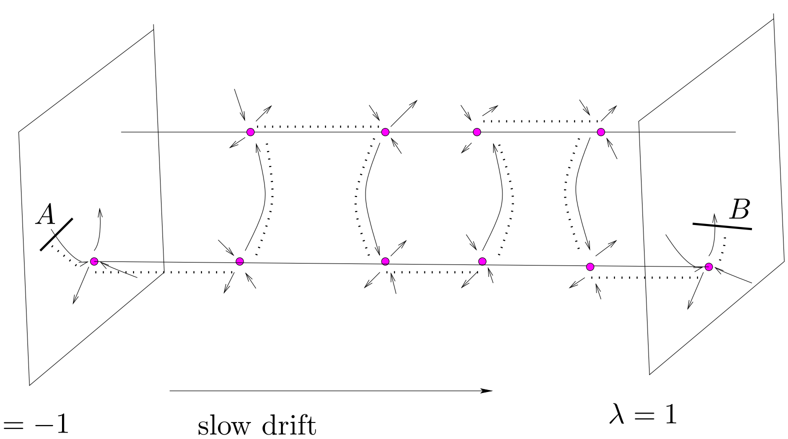

Figure 1: The structure of a solution to boundary value problem on $[-1, 1]$ with $n = k = l = 1$.

An esssential difficulty is that in general the singular flow φ^0 does not admit the desired isolating neighborhood. This led Conley [1] to introduce the concept of a *singular isolating neighborhood*; that is a compact set N which is an isolating neighborhood for φ^ϵ, $\epsilon > 0$. To understand the structure of maximal invariant set $S^\epsilon := \text{Inv}(N, \varphi^\epsilon)$ in N for $\epsilon > 0$, we shall make use of the cohomological Conley index of S^ϵ which is denoted by $CH^*(S^\epsilon, \varphi^\epsilon)$. However, the computation of the index is done using the singular flow φ^0. This requires the idea of a *singular index pair* [3]; that is, a pair of compact sets $\mathbf{L} \subset \mathbf{N}$ such that

$$CH^*(S^\epsilon, \varphi^\epsilon) \cong H^*(\mathbf{N}, \mathbf{L}).$$

The key idea is to construct a singular isolating neighborhood and its singular index pair around the concatenation using the relatively simple flow ψ^Λ, compute its Conley index [11] and deduce the existence of invariant sets of φ^ϵ for small ϵ. The construction of the singular isolating neighborhood and the singular index pair as well as the computation the Conley index is done in a modular fasion, that is, by using only compact neighborhoods of the connecting orbits and and compact neighborhoods of segments of the branches of invariant sets for φ^0.

We will preserve this modularity, but at this point our approach diverges from that of [3]. We are not interested in invariant sets in the isolating neighborhood (in fact for $\epsilon > 0$, the system (3) has no invariant sets), but rather a solution starting at $\lambda = -1$ satisfying first boundary condition (2), lying in the constructed singular isolating neighborhood over the interval Λ and satisfying second boundary condition at $\lambda = 1$. To do this we encode the boundary conditions into algebraic information. We assume that each boundary condition generates a certain cohomology or homology group in the intersection of the singular isolating neighborhood and the corresponding slice $R^{2n} \times \{-1\}$ (or $R^{2n} \times \{1\}$, respectively). These groups are related to the Conley index of the whole singular isolating neighborhood. We will show that for small ϵ the flow φ^ϵ induces a homotopy which relates the generator of the cohomology group generated by the boundary condition at $\lambda = 1$ to the generator of a cohomology group at $\lambda = -1$. This generator has a nontrivial pairing with the generator of the homology group generated by the boundary condition at $\lambda = 1$. Since the pairing is nontrivial, the supports of these generators must have a nonempty intersection, which implies the existence of a solution of the boundary value problem lying in the singular isolating neighborhood. For related ideas concerning intersection pairing of Conley indices see [5, 6].

We describe now the construction of a singular isolating neighborhood $\mathbf{N}$, consisting of pieces along branches of invariant sets and along connecting orbits between branches. Recall that a *Morse decomposition*

$$\mathcal{M}(S) = \{M(p) \mid p \in (\mathcal{P}, >)\}$$

of an isolated invariant set S is a finite collection of disjoint compact invariant subsets $M(p)$, called *Morse sets*, indexed by a partially ordered set $(\mathcal{P}, >)$, with the property that; if $x \in S \setminus \bigcup_{p \in P} M(p)$, then there exist $q > p$ such that the alpha limit set of x is contained in $M(q)$ and the omega limit set of x is contained in $M(p)$.

The segments of $\mathbf{N}$ around the branches of Morse sets are the simplest to define. We adapt definitions from [3] to our case.

Definition 1.1 $\mathcal{T} \subset \mathbf{R}^{2n} \times \mathbf{R}$ is a *tube* if there exists an interval $[a, b]$ such that $\mathcal{T} \subset \mathbf{R}^{2n} \times [a, b]$ and $\mathcal{T}$ is an isolating neighborhood for the parameterized flow

$$\begin{array}{rcl} \psi^{\mathcal{T}} : \mathbf{R} \times \mathbf{R}^{2n} \times [a, b] & \to & \mathbf{R}^{2n} \times [a, b], \\ (t, x, \lambda) & \mapsto & (\psi_\lambda(t, x), \lambda). \end{array}$$

We now turn to the neighborhoods which connect tubes along different branches of Morse sets. Each inner layer of a solution of the boundary value problem for small

ϵ should lie close to a connecting orbit between two Morse sets on different branches. The Conley index theory provides a variety of techniques for proving the existence of connecting orbits. We will use *topological transition matrices*.

A Morse decomposition of a parameterized flow $\psi^\Lambda : \mathbf{R} \times \mathbf{R}^{2n} \times \Lambda \to \mathbf{R}^{2n} \times \Lambda$, is said to *continue over* Λ if there is an isolated invariant set $S = \mathrm{Inv}(N, \psi^\Lambda)$ with a Morse decomposition $\mathcal{M}(S) = \{M(p) \mid p \in (\mathcal{P}, >)\}$. Observe that if one defines

$$S_\lambda := S \cap (\mathbf{R}^{2n} \times \{\lambda\}),$$

then S_λ is an isolated invariant set for ψ^λ. Similarly, $\{M_\lambda(p) \mid p \in (\mathcal{P}, >)\}$ is a Morse decomposition for S_λ. Since Morse sets are isolated invariant sets, $CH^*(M_\lambda(p))$ is defined. Furthermore, the index of each Morse set remains constant over Λ. Let $\lambda_0, \lambda_1 \in \Lambda$ and assume that

$$S_{\lambda_i} = \bigcup_{p \in \mathcal{P}} M_{\lambda_i}(p) \quad i = 0, 1.$$

Then, there exists a lower triangular (with respect to the order $>$) degree 0 isomorphism

$$T^{\lambda_1, \lambda_0} : \bigoplus_{p \in \mathcal{P}} CH^*(M_{\lambda_0}(p)) \to \bigoplus_{p \in \mathcal{P}} CH^*(M_{\lambda_1}(p))$$

called a topological transition matrix (see [9, 10]). Roughly, if the (p, q) off-diagonal entry of T^{λ_1, λ_0} is non-zero, then for some parameter value $\lambda \in (\lambda_0, \lambda_1)$ there exists a connecting orbit between $M_\lambda(p)$ and $M_\lambda(q)$. In order to insure the existence of topological transition matrices we introduce the following neighborhoods of the connecting orbits.

Definition 1.2 A set $\mathcal{B} \subset \mathbf{R}^{2n} \times \mathbf{R}$ is a *box* if:

1. There exists an interval $[c, d]$ such that $\mathcal{B} \subset \mathbf{R}^{2n} \times [c, d]$ and $\mathcal{B}$ is an isolating neighborhood for the parameterized flow $\psi^{\mathcal{B}}$ defined by

$$\begin{array}{rcl} \psi^{\mathcal{B}} : \mathbf{R} \times \mathbf{R}^{2n} \times [c, d] & \to & \mathbf{R}^{2n} \times [c, d], \\ (t, x, \lambda) & \mapsto & (\psi^\lambda(t, x), \lambda). \end{array}$$

2. Let $S(\mathcal{B}) := \mathrm{Inv}(\mathcal{B}, \psi^{\mathcal{B}})$. There exists a Morse decomposition

$$\mathcal{M}(S(\mathcal{B})) := \{M(p, \mathcal{B}) \mid p = 1, \dots, P_{\mathcal{B}}\},$$

with the usual ordering on the integers as an admissible ordering. Let $\mathcal{B}_\lambda = \mathcal{B} \cap (\mathbf{R}^{2n} \times \{\lambda\})$, $S_\lambda(\mathcal{B}) := \mathrm{Inv}(\mathcal{B}_\lambda, \psi^\lambda)$ and let $\{M_\lambda(p, \mathcal{B}) \mid p = 1, \dots, P_{\mathcal{B}}\}$ be the corresponding Morse decomposition of $S_\lambda(\mathcal{B})$. Then

$$S_c(\mathcal{B}) := \bigcup_{p=1}^{P_{\mathcal{B}}} M_c(p, \mathcal{B}) \quad \text{and} \quad S_d(\mathcal{B}) := \bigcup_{p=1}^{P_{\mathcal{B}}} M_d(p, \mathcal{B}).$$

3. There are isolating neighborhoods $V(p,\mathcal{B})$ for $M(p,\mathcal{B})$ such that

$$V(p,\mathcal{B}) \subset \mathcal{B} \quad \text{and} \quad V(p,\mathcal{B}) \cap V(q,\mathcal{B}) = \emptyset$$

   for $p \neq q$ and for every $\lambda \in [c,d]$

$$V_\lambda(p,\mathcal{B}) \subset \operatorname{int}(\mathcal{B}_\lambda)$$

Notice that Definition 1.2(2) implies that there are no connecting orbits between the Morse sets at the parameter values c and d, and by the construction, the sets $S_c(\mathcal{B})$ and $S_d(\mathcal{B})$ are related by continuation. Let

$$T^{\mathcal{B}}(P,1) : CH^*(M_d(1,\mathcal{B})) \to CH^*(M_c(P,\mathcal{B}))$$

denote the $(P,1)$-entry of the matrix $T^{\mathcal{B}}$.

It is important to realize that the notions of Morse sets and topological transtion matrix are very general. As an example consider the nonlinearity f in (3) of the form

$$f(u,\lambda) = q(\lambda) + (u^2 - 25)(u^2 - 16)(u^2 - 9)$$

where $g(\lambda) = -600\lambda^2 + 400$, see Figure 2. As λ varies over $\Lambda = [-1,1]$ there are two Morse sets $M(1)$ and $M(2)$ that continue across Λ; one that consists of $1,2$ or 3 negative equilibria and one that consists of $1,2$ or 3 positive equilibria. We can define a box $\mathcal{B}$ over the interval $[-1,1]$ with a Morse decomposition consisting of $M(1)$ and $M(2)$. If the corresponding topological transition matrix has a nontrivial $(1,2)$ entry, then there is a connecting orbit from the Morse set $M(1)$ to the Morse set $M(2)$. This topological approach does not provide (but also does not require) a more detailed information about the precise parameter value where the connection occurs, nor which one of the (potentialy) three equilibria is involved in this connection.

Finally, in order to construct a singular isolating neighborhood, these boxes and tubes must to be related in a consistent manner. The primary requirement is that the tubes and boxes overlap at the appropriate Morse sets. To simplify the notation we let $P_i = P_{\mathcal{B}_i}$ and $M(p,i) := M(p,\mathcal{B}(i))$.

Definition 1.3 A set of tubes $\{\mathcal{T}(i) \mid i = 1,\ldots,I+1\}$ and boxes $\{\mathcal{B}(i) \mid i = 1,\ldots,I\}$ forms a *tubes and boxes collection over* $\Lambda = [-1,1]$ *(a TB collection)* if the following compatibility conditions are satisfied (see Figure 3):

1. for $i = 1,\ldots,I$
   (a) $\mathcal{T}(i) \cap (\mathbf{R} \times [c_i,d_i]) \subset V(1,\mathcal{B}(i))$ and $\mathcal{T}(i) \cap \mathcal{B}(i)$ isolates $M(1,i)$.
   (b) $\mathcal{T}(i+1) \cap (\mathbf{R} \times [c_i,d_i]) \subset V(P_i,\mathcal{B}(i))$ and $\mathcal{T}(i+1) \cap \mathcal{B}(i)$ isolates $M(P_i,i)$.
2. for $i = 1,\ldots,I$

$$b_{i+1} = d_i \qquad \text{and} \qquad a_i = c_i$$

   where a, b, c, and d are as in Definitions 1.1 and 1.2.
3. If $i \neq j$, then $\mathcal{B}(i) \cap \mathcal{B}(j) = \emptyset$.

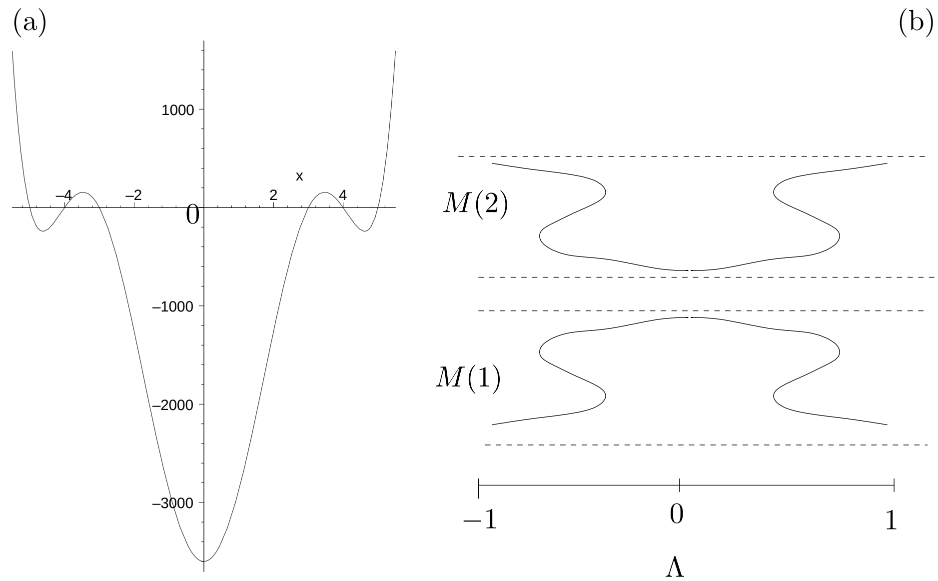

Figure 2: (a) The graph of the polynomial $(x^2-25)(x^2-16)(x^2-9)$. (b) The bifurcation diagram of $f(u,\lambda)$.

4. We require that the tubes cover the whole interval Λ

$$\bigcup_{i=1}^{I+1} [a_i, b_i] = \Lambda.$$

We introduce one final bit of notation before stating some of the results of this paper. We start by reviewing some basic facts about continuation. To simplify the presentation we begin by choosing the parameter space $[0,1]$, and assume that the Morse decomposition $\mathcal{M}(S) := \{M(p) \mid p \in (\mathcal{P}, >)\}$ continues over all of $[0,1]$. Then by Theorem 6.7 [12] there are isomorphisms

$$F^*_{1,0}(p) : CH^*(M_0(p)) \to CH^*(M_1(p)).$$

Fix a set of generators $\mathcal{G}_0$ of $\bigoplus_{p\in\mathcal{P}} CH^*(M_0(p))$. Define the generators of $\bigoplus_{p\in\mathcal{P}} CH^*(M_1(p))$ to be

$$\mathcal{G}_1 := \bigoplus_{p\in\mathcal{P}} F^*_{1,0}(p)(\mathcal{G}_0).$$

With this identification, $\bigoplus_{p\in\mathcal{P}} F^*_{1,0}(p)$ takes the form of the identity matrix.

From this point on we shall assume that this identification has been made and we refer to it as the *natural Morse continuation identification.*

Following in the same spirit, observe that each tube $\mathcal{T}(i)$ defines a continuation between the invariant sets $M_{d_{i+1}}(1, i+1)$ and $M_{c_i}(P_i, i)$. Thus $CH^*(M_{d_{i+1}}(1, i+1)) \cong CH^*(M_{c_i}(P_i, i))$. Since the adjacent boxes $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ are disjoint, we can use this continuation to choose basis for these linear spaces in such a way that the matrix representing the continuation is the identity matrix. We shall refer to this choice of bases as the *natural tube continuation identification.*

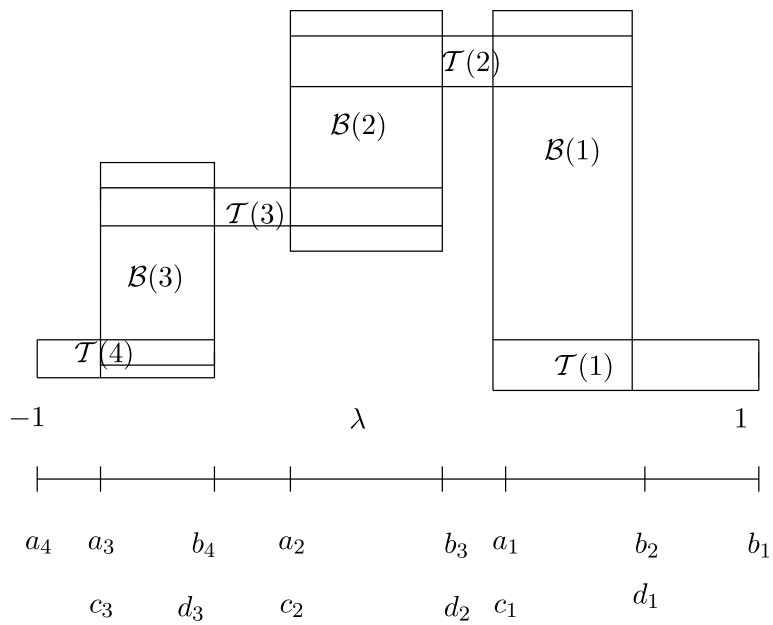

Figure 3: A TB collection over the interval $[-1, 1]$.

For a TB collection we set

$$\mathcal{N} := \bigcup_{i=1}^{I+1} \mathcal{T}(i) \cup \bigcup_{i=1}^{I} \mathcal{B}(i). \tag{5}$$

Furthermore, let

$$T^i : \bigoplus_{p=1}^{\mathcal{P}_i} CH^*(M_{d_i}(p,i)) \to \bigoplus_{p=1}^{\mathcal{P}_i} CH^*(M_{c_i}(p,i))$$

denote the transition matrix associated with the box $\mathcal{B}(i)$ and let

$$T^i(P_i, 1) : CH^*(M_{d_i}(1,i)) \to CH^*(M_{c_i}(P_i,i)) \tag{6}$$

denote the corresponding entry. Finally, we define a map

$$\Theta^* := \Theta(I) = T^I(P_I, 1) \circ T^{I-1}(P_{I-1}, 1) \circ \ldots \circ T^2(P_2, 1) \circ T^1(P_1, 1). \tag{7}$$

As stated, this definition obviously makes no sense since $CH^*(M_{c_i}(P_i,i)) \neq CH^*(M_{d_i}(1, i+1))$. However, these spaces are identified by a natural tube continuation identification.

Theorem 1.4 *Consider a system (1) with boundary conditions (2). Assume that (3) with $\epsilon = 0$ has saddle points $(u^-(\lambda), v^-(\lambda), \lambda)$ for all $\lambda \in [-1, b_{I+1}]$ and $(u^+(\lambda), v^+(\lambda), \lambda)$ for all $\lambda \in [a_1, 1]$ each of which is hyperbolic under flow ψ^λ. Let $U^+ := (u^+(1), v^+(1), 1)$ and $U^- := (u^-(-1), v^-(-1), -1)$. We denote by $W^s(U^\pm)$ the stable and $W^u(U^\pm)$ the unstable manifolds of $U^\pm$ and we let* $\dim\, A = k, \dim\, B = l$. *Further assume that*

1. *$A \cap W^s(U^-) \neq \emptyset$, $B \cap W^u(U^+) \neq \emptyset$ and the intersections are transversal;*
2. *$k + l = 2n$ and $\dim W^s(U^-) = l$, $\dim W^u(U^+) = k$;*
3. *for $\epsilon = 0$ system (3) admits a TB collection over Λ such that the corresponding map $\Theta^* \neq 0$.*

Then there is an $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*]$, (1) and (2) have a solution lying in $\mathcal{N}$ for all $x \in [a_I, b_2]$.

Remark 1.5 The assumption that A intersects transversaly $W^s(U^-)$ and B intersects transversaly $W^u(U^+)$ is stronger then neccessary. As will be clear from proof of Lemma 2.2 all we need is that A generates a local homology group

$$H_k(B_\eta, B_\eta \setminus B_\delta(W^s(U^-))) \cong Z$$

where B_η is a small neighborhood of the intersection of $A \cap W^s(U^-)$ and $B_\delta(W^s(U^-))$ is a δ-neighborhood (collar) of $W^s(U^-)$, $\delta << \eta$. Similarly, B needs to generate a local cohomology group

$$H^l(B_\eta, B_\eta \setminus B_\delta(W^s(U^+))) \cong Z.$$

We want to emphasize that our assumptions are topological in nature, and we formulate stronger geometrical assumptions only for the sake of clarity of exposition.

2 Singular isolating neighborhood and index pair

Let $\varphi : \mathbf{R} \times X \to X$ be a flow on a locally compact space X. Given a set N of S its *immediate exit* and *entrance sets* are defined, respectively, as follows

$$N^- := \{x \in N \mid \varphi([0,t], x) \not\subset N \text{ for all } t > 0\},$$

$$N^+ := \{x \in N \mid \varphi([t,0], x) \not\subset N \text{ for all } t < 0\}.$$

We will consistently use the script letters $\mathcal{N}, \mathcal{T}$ and $\mathcal{B}$ to denote parts of singular isolating neigborhood. The bold face letters $\mathbf{N}, \mathbf{T}$ and $\mathbf{B}$ will be reserved for the corresponding parts of the singular index pairs.

Lemma 2.1 *[3, Propositions 3.6, 3.7] Consider system (1) with boundary conditions (2). Assume that for $\epsilon = 0$ the system (3) admits a TB collection over Λ with singular isolating neighborhood $\mathcal{N}$ as defined by (5).*

Then, there are sets $(\mathbf{N}, \mathbf{L})$ with

$$\mathbf{N} := \left(\bigcup_{i=1}^{I+1} \mathbf{T}(i) \right) \bigcup \left(\bigcup_{i=1}^{I} \mathbf{B}(i) \right),$$

consisting of tubes $\mathbf{T}(i)$ and boxes $\mathbf{B}(i)$, such that $(\mathbf{N}, \mathbf{L} \cap W^u_{\mathbf{T}(1)}(U^+)$ is a singular index pair for invariant the set $\mathrm{Inv}\,(\mathcal{N}, \varphi^\epsilon)$. Furthermore, the tubes $\mathbf{T}(i)$ and boxes $\mathbf{B}(i)$ are isolating blocks under the parameterized flow φ^0.

Proof. This follows directly from results [3, Section 3], in particular Propositions 3.6 and 3.7. For the completeness we review the construction of the set $\mathbf{L}$.

Let $\mathbf{T}^-(i)$ be the immediate exit set from tube $\mathbf{T}(i)$ and let $\mathbf{B}^-(i)$ be the immediate exit set from the box $\mathbf{B}(i)$. Define

$$\mathbf{N}^- = \left(\left(\bigcup_{i=1}^{I+1} \mathbf{T}^-(i) \right) \cup \left(\bigcup_{i=1}^{I} \mathbf{B}^-(i) \right) \right) \setminus \left(\bigcup_{i=1}^{I} \{ \mathbf{B}(i) \cap (\mathbf{T}^-(i) \cup \mathbf{T}^-(i+1)) \} \right),$$

and

$$\mathbf{L} := \rho(cl(\mathbf{N}^-), \mathbf{N}, \varphi^0) \cup \left(\bigcup_{i=1}^{I} \bigcup_{p=2}^{P_i} W^u_{\mathbf{B}(i)} \left(M_{d_i}(p,i) \right) \right). \tag{8}$$

The set $\rho(cl(\mathbf{N}^-), \mathbf{N}, \varphi^0)$ is the *push-forward* of the set $\mathbf{N}^-$ by the flow φ^0 in the set $\mathbf{N}$:

$$\rho(cl(\mathbf{N}^-), \mathbf{N}, \varphi^0) := \bigcup_{z \in \mathbf{N}^-} \bigcup_{t \in [0, t(z)]} \varphi^0(t, z)$$

where $t(z)$ is the maximal time such that $\varphi^0(z, t(z)) \in \mathbf{N}$. By results in [3, Section 3] the pair $(\mathbf{N}, \mathbf{L} \cap W^u_{\mathbf{T}(1)}(U^+))$ is a singular index pair for the maximal invariant set in $\mathcal{N}$. □

By [11, Theorem 1.15] the singular index pair $(\mathbf{N}, \mathbf{L} \cap W^u_{\mathbf{T}(1)}(U^+))$ is an index pair for invariant set Inv$\mathcal{N}$ under flow φ^ϵ for all sufficiently small ϵ. This invariant set is clearly empty and thus $H^*(\mathbf{N}, \mathbf{L} \cap W^u_{\mathbf{T}(1)}(U^+)) = 0$.

In the following discussion a pair $(\mathbf{N}, \mathbf{L})$ rather than $(\mathbf{N}, \mathbf{L} \cap W^u_{\mathbf{T}(1)}(U^+)$ plays the key role. To accomodate the presence of the boundary conditions and in order to encode these boundary conditions in cohomology, we need to make careful changes to the construction of the tubes $\mathbf{T}(1)$ and $\mathbf{T}(I+1)$ from [3].

Lemma 2.2 *Consider system (1) with boundary conditions (2). Assume that (3) with* $\epsilon = 0$ *has saddle points* $(u^-(\lambda), v^-(\lambda), \lambda)$ *for all* $\lambda \in [-1, b_{I+1}]$ *and* $(u^+(\lambda), v^+(\lambda), \lambda)$ *for all* $\lambda \in [a_1, 1]$ *each of which is hyperbolic under flow* ψ^λ*. Let* $U^+ := (u^+(1), v^+(1), 1)$ *and* $U^- := (u^-(-1), v^-(-1), -1)$*. We denote by* $W^s(U^\pm)$ *the stable and* $W^u(U^\pm)$ *the unstable manifolds of* $U^\pm$*. Let* $k :=$ dim A *and* $l :=$ dim B*. Further assume the following:*

1. $A \cap W^s(U^-) \neq \emptyset$*,* $B \cap W^u(U^+) \neq \emptyset$ *and the intersections are transversal;*
2. $k + l = 2n$ *and* $\dim W^s(U^-) = l$*,* $\dim W^u(U^+) = k$*;*
3. *for* $\epsilon = 0$ *system (3) admits a TB collection over* Λ*.*

Then there are sets $(\mathbf{N}, \mathbf{L})$ *that satisfy conclusions of Lemma 2.1 with the additional properties:*

1. *the set* $\mathbf{T}_{-1}$ *is connected;*
2. *there are generators* $\alpha \in H_k(\mathbf{T}_{-1}, \mathbf{T}^-_{-1})$ *and* $\beta \in H^l(\mathbf{T}_1, \mathbf{T}^-_1)$ *with support*

$$supp(\alpha) = A \cap \mathbf{T}_{-1}, \quad \text{and} \quad supp(\beta) = B \cap \mathbf{T}_1,$$

where $T^-_{-1} := \mathbf{L} \cap (\mathbf{R}^{2n} \times \{-1\})$ *and* $T^-_1 := \mathbf{L} \cap (\mathbf{R}^{2n} \times \{1\})$*.*

Proof. By Lemma 2.1 (and thus ([3])) we can replace isolating neighborhoods $\mathcal{T}(i)$ and $\mathcal{B}(i)$ in TB collection by isolating blocks $\mathbf{T}(i)$ and $\mathbf{B}(i)$ respectively, such that $\{\mathbf{T}(i) \,|\, i = 1, \ldots, I+1\}$ and $\{\mathbf{B}(i) \,|\, i = 1, \ldots, I\}$ form a TB collection. We now describe a modification of the tubes $\mathbf{T}(1)$ and $\mathbf{T}(I+1)$ to accomodate the sets A and B. We will perform the construction for $\mathbf{T}(I+1)$ in the neighborhood of $\lambda = -1$. The construction for $\mathbf{T}(1)$ near $\lambda = 1$ is analogous with the flow φ^0 replaced by the inverse flow $(\varphi^0)^{-1}$.

We first describe our construction in an idealized case and then show that the general case can be transformed to the ideal case by a change of variables. In the ideal situation

1. the stable manifold W^s and the unstable manifold W^u of the hyperbolic saddle $(u^-(\lambda), v^-(\lambda), \lambda)$ is descibed by equations $x = 0$ and $y = 0$, respectively, for all $\lambda \in [-1, -1+\delta_0], \delta_0 < b_{I+1}$. Here, by assumption, $x \in \mathbf{R}^k$, $y \in \mathbf{R}^l$ and $k+l = 2n$.
2. There is a constant η such that the flow φ^0 is linear in $[-\eta, \eta]^{2n} \times [-1, b_{I+1}]$ and $[-\eta, \eta]^{2n} \times [-1, b_{I+1}] = \mathbf{T}(I+1)$.
3. There are constants $\zeta < \eta$ and $M > \eta$ such that the x component of the flow φ^0 in
$$D(\zeta) := \{[-\zeta, \zeta]^k \times [-M, M]^l \times [-1, -1+\delta_0]\} \setminus \mathbf{T}(I+1)$$
vanishes and $A \cap W^s(U^-) \subset D(\zeta)$, see Figure 4.a.

Observe that the first and second condition follows by change of variables from the Hartman-Grobman Theorem [4] in sufficently small neigborhood of $(u^-(\lambda), v^-(\lambda), \lambda)$ and the replacement of an old isolating block $\mathbf{T}(I+1)$ by a new isolating block of the form $[-\eta, \eta]^{2n} \times [-1, b_{I+1}]$. The third condition follows from the Flow Box Theorem.

We construct the new set $\bar{\mathbf{T}}(I+1)$ in several steps.

1. Let
$$Y(\zeta) = \rho(D(\zeta), D(\zeta) \cup \mathbf{T}(I+1), \varphi^0)$$
be the push-forward of $D(\zeta)$ in $D(\zeta) \cup \mathbf{T}(I+1)$ under the parameterized flow φ_0, see Figure 4.b.
2. We now "shave" the set $Y(\zeta)$. Consider the part of the boundary of $Y(\zeta)$ defined by
$$Q(\zeta) := \bigcup_{x \in D(\zeta) \cap \partial[-\zeta,\zeta]^k \times [-M,M]^l \times [-1,-1+\delta_0]} \varphi^0(x, [0, T_x]),$$
where $T_x := sup\{t \mid \varphi^0(x,t) \cap Y(\zeta) \neq \emptyset\}$ is the exit time and ∂X denotes the boundary of the set X. Take $\zeta' < \zeta$ and shave the set $Y(\zeta)$ between the sets $Q(\zeta')$ and $Q(\zeta)$ in such a way that the shaved boundary is a strict exit set. We call this new, shaved set $W(\zeta)$.
3. Now we will take into account the full flow φ^ϵ, not just the flow φ^0. The flow φ^ϵ has a constant drift in the positive λ direction. Notice that by assumption all points $(x, y) \in W(\zeta) \setminus \mathbf{T}(I+1)$ enter $\mathbf{T}(I+1)$ in finite time under the flow φ^0. Let T be maximum of these times. In time T the flow φ^ϵ drifts in the λ direction the amount ϵT. Fix ϵ_1 such that $\delta_1 := \epsilon_1 T < \delta_0/2$. Let
$$G(\zeta, \epsilon_1) := \rho(W(\zeta) \cap (\mathbf{R}^{2n} \times [-1, -1+\delta_1]), W(\zeta), \varphi^{\epsilon_1})$$

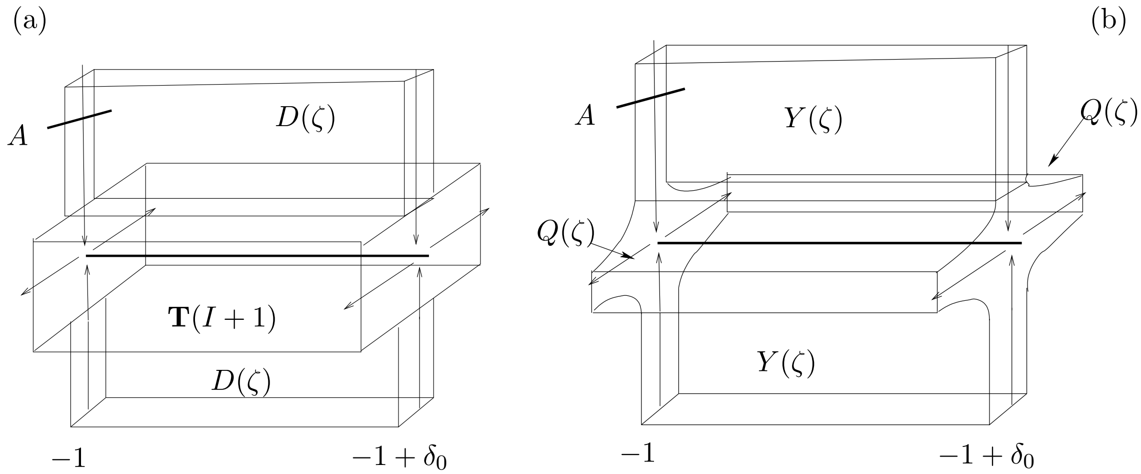

Figure 4: (a) The sets $D(\zeta)$ and $\mathbf{T}(I+1)$ for $n = 1$. (b) The sets $Y(\zeta)$ and $Q(\zeta)$. The set $Y(\zeta)$ is a push-forward of the set $D(\zeta)$. Front and back parts of $Y(\zeta)$ form a set $Q(\zeta)$.

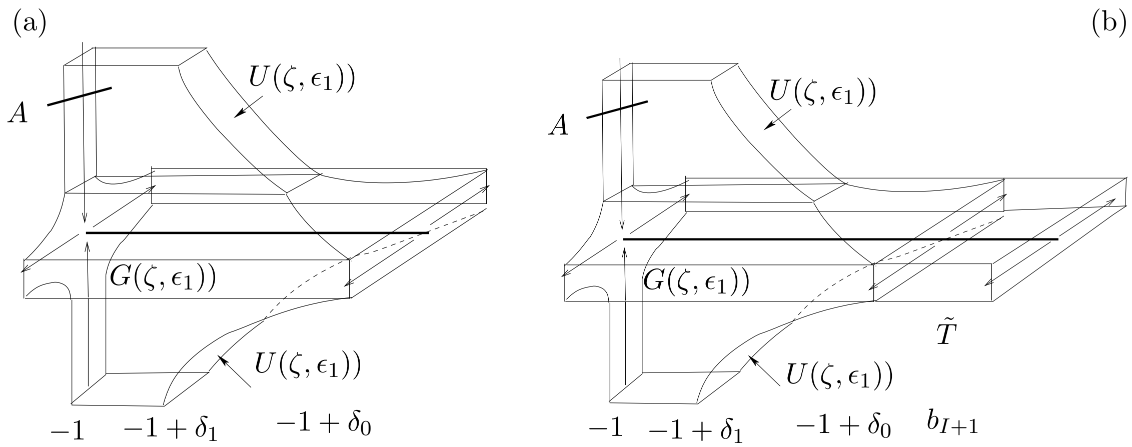

Figure 5: (a) The set $G(\zeta,\epsilon_1))$. (b) The set $\mathbf{T}(I+1)$ is a union of $W(\zeta,\epsilon_1)$ and $\tilde{T}_{[-1+\delta_1,b_{I+1}]}$.

be the push-forward of the restriction of $W(\zeta)$ to $\lambda \in [-1,-1+\delta_1]$, see Figure 5.a.

4. The set

$$U(\zeta,\epsilon_1) := \bigcup_{x\in D(\zeta)\cap[-\zeta,\zeta]^k\times[-M,M]^l\times\delta_1} \varphi^{\epsilon_1}(x,[0,T_x^{\epsilon_1}])$$

is an ϵ_1-flow invariant part of the boundary of $G(\zeta,\epsilon_1)$. By possibly choosing a smaller ϵ_1, we perturb this part of boundary of $G(\zeta,\epsilon_1)$ so that the new boundary is a strict entrance set for φ^ϵ for all $\epsilon < \epsilon_1$. We call this resulting shaved set $W(\zeta,\epsilon_1)$.

5. Let

$$\tilde{T} = [-\eta,\eta]^k \times [-\eta_1,\eta_1] \times \ldots \times [-\eta_l,\eta_l] \times [-1+\delta_1, b_{I+1}]$$

be an isolating block over interval $[-1+\delta_1, b_{I+1}]$ where $\eta_1,\ldots,\eta_l$ are selected in such a way that

$$\tilde{T} \cap (\mathbf{R}^{2n} \times \{-1+\delta_1\}) = W(\zeta, \epsilon_1) \cap (\mathbf{R}^{2n} \times \{-1+\delta_1\},$$

see Figure 5.b. Finally, we set

$$\bar{\mathbf{T}}(I+1) := W(\zeta, \epsilon_1) \cup \tilde{T}.$$

With the construction of the new isolating block $\bar{\mathbf{T}}(I+1)$ complete, we will prove that it has the advertized properties. In order to simplify the notation we drop the bar from $\bar{\mathbf{T}}(I+1)$ in what follows.

It is easy to see that $\mathbf{T}(I+1)$ still isolates $(0,0,\lambda)$ for $\lambda \in [-1, b_{I+1}]$ and for all $\epsilon \in (0, \epsilon_1]$. The set $\mathbf{T}_{-1}$ has the form

$$\mathbf{T}_{-1} = [-\eta, \eta]^{2n} \cup W_{-1}(\zeta, \epsilon_1) = [-\eta, \eta]^{2n} \cup W_{-1}(\zeta),$$

since the change from $W(\zeta)$ to $W(\zeta, \epsilon_1)$ has not affected the set $W_{-1}(\zeta) = W(\zeta) \cap (\mathbf{R}^{2n} \times \{-1\})$. By definition of the set $W(\zeta)$ the set $\mathbf{T}_{-1}$ is homotopically equivalent to set $[-\eta, \eta]^{2n}$. Thus $\mathbf{T}_{-1}$ is connected. The immediate exit set of $\mathbf{T}_{-1}$ is

$$Q(\zeta) \cup \left((\partial[-\eta, \eta]^k) \times [-\eta, \eta]^l \right).$$

Again, by construction, the set $Q(\zeta)$ is homotopically equivalent to the set $(\partial[-\eta, \eta]^k) \times [-\eta, \eta]^l$ via the flow φ^0. Therefore

$$H_*(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-) \cong H_*([-\eta, \eta]^{2n}, (\partial[-\eta, \eta]^k) \times [-\eta, \eta]^l)$$

which gives

$$H_*(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-) \cong \begin{cases} Z & \text{if } * = k \\ 0 & \text{otherwise .} \end{cases}.$$

Since ζ in the construction of $W(\zeta)$ can be choosen arbitrarily small and A intersects $W^s(U^-)$, we can choose ζ small enough so that

$$A \cap \partial \mathbf{T}_{-1} \subset Q(\zeta) \subset \mathbf{T}_{-1}^-. \tag{9}$$

It follows that there is a generator $\alpha \in H_k(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-)$ with support $supp(\alpha) = A \cap \mathbf{T}_{-1}$ Similar construction, using the inverse flow $(\varphi^0)^{-1}$, applies to set $\mathbf{T}(1)$. Here the set analogous to $Q(\zeta)$ will be strict exit set under inverse flow, which means it will be strict entrance set under φ^0. Therefore, instead of (9) we will have

$$(B \cap \partial \mathbf{T}(1)) \cap \mathbf{T}_1^- = (B \cap \partial \mathbf{T}_1) \cap \mathbf{T}_1^- = \emptyset,$$

which implies that there is a cohomology generator $\beta \in H^l(\mathbf{T}_1, \mathbf{T}_1^-)$ with support $supp(\beta) = B \cap \mathbf{T}_1$, see Figure 6. □

We observe that by the natural tube continuation (Section 2)

$$CH^*(M_{d_1}(1,1)) \cong H^*(\mathbf{T}_1, \mathbf{T}_1^-) \quad \text{and} \quad CH^*(M_{c_I}(P_I, I)) \cong H^*(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-).$$

With this observation we have

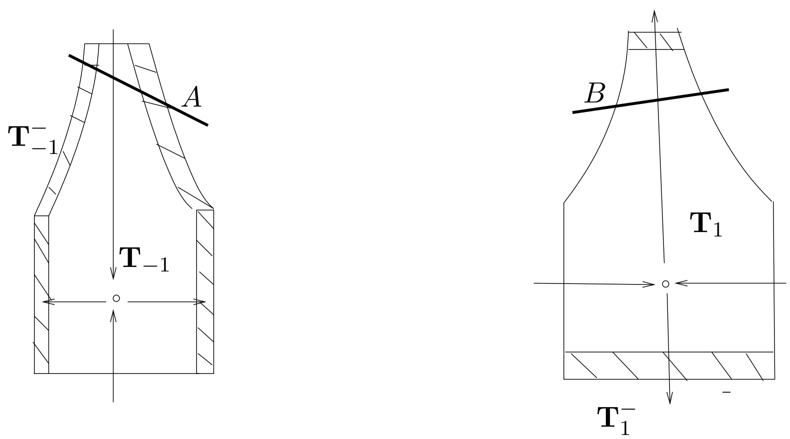

Figure 6: Selection of $\mathbf{T}_{-1}$ and $\mathbf{T}_1$ in relation to boundary conditions A and B.

Proposition 2.3 *[3, Proposition 4.1 and 4.6] There is an inclusion induced isomorphism*

$$\Psi : H^*(\mathbf{N}, \mathbf{L}) \to H^*(\mathbf{T}_1, \mathbf{T}_1^-).$$

This isomorphism can be related to the map Θ^* introduced in (7).

Proposition 2.4 *[3, Proposition 4.8] Let $\xi : (\mathbf{T}_{-1}, \mathbf{T}_{-1}^-) \hookrightarrow (\mathbf{N}, \mathbf{L})$ be the inclusion. Then*

$$\xi^* : H^*(\mathbf{N}, \mathbf{L}) \to H^*(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-)$$

is equal to the composition map

$$\Theta^* \circ \Psi : H^*(\mathbf{N}, \mathbf{L}) \to H^*(\mathbf{T}_1, \mathbf{T}_1^-) \to H^*(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-).$$

In other words, the following diagram commutes:

$$\begin{tikzcd} H^*(\mathbf{N}, \mathbf{L}) \arrow[r, "\cong"] \arrow[dr, "\xi^*"'] & H^*(\mathbf{T}_1, \mathbf{T}_1^-) \arrow[d, "\Theta^*"] \\ & H^*(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-) \end{tikzcd}.$$

Lemma 2.5 *The cap product*

$$H_*(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-) \frown H^*(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-) \to H_0(\mathbf{T}_{-1}). \tag{10}$$

is well defined.

Proof. Since

$$H^*(\mathbf{T}_{-1}^-, \mathbf{T}_{-1}^- \cap \emptyset) = H^*(\mathbf{T}_{-1}^-) = H^*(\mathbf{T}_{-1}^- \cup \emptyset, \emptyset)$$

by [2, Proposition III.8.1], the triad $(\mathbf{T}_{-1}; \mathbf{T}_{-1}^-, \emptyset)$ is excisive. As a consequence, by [2, VII 12.1], the cap product (10) is well defined. $\square$

3 ϵ-flow defined homotopy

In this section we introduce a φ^ϵ defined homotopy which is the crucial tool in the proof of Theorem 1.4. We first outline the argument of the proof. By Lemma 2.2, B is the support of the generator β of $H^l(\mathbf{T}_1, \mathbf{T}_1^-)$. By Proposition 2.3 $H^l(\mathbf{T}_1, \mathbf{T}_1^-)$ is isomorphic to $H^l(\mathbf{N}, \mathbf{L})$. We will show that this isomorphism can be realized by a φ^ϵ-induced map $P^{\epsilon *}$. In fact, a slight modification of the set $\mathbf{L}$ is required for this step. We will show that since $\Theta^* \neq 0$, $\gamma := \xi^*(P^{\epsilon *}(\beta))$ is nontrivial in $H^l(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-)$. Once this is established, we use the natural pairing in Lemma 2.5 to show that $\gamma \frown \alpha$ is a nontrivial generator in $H_0(\mathbf{T}_{-1})$. This in turn implies that

$$supp(\gamma) \cap supp(\alpha) \neq \emptyset.$$

As we will show this is equivalent to solving boundary value problem (1),(2).

Consider the flow defined map

$$p^\epsilon : [0, T] \times \mathbf{N} \to \mathbf{N}$$

for small $\epsilon > 0$ and $T > 2/\epsilon$, which maps $x \in \mathbf{N}$ to the point where trajectory $\varphi^\epsilon(x, t)$ leaves N. We would like to show that for sufficiently small ϵ

1. the map p^ϵ is continuous
2. $p^\epsilon(\mathbf{N}, T) \subset \mathbf{L} \cup \mathbf{T}_1$.

However, for the standard construction of a singular index pair (see [3]) neither of these statements is true. A modification of the set $\mathbf{L}$ to a set $\mathbf{L}_\delta$ and a modification of the definition of p^ϵ will be needed to achieve this goal.

Recall that by (8) the exit set $\mathbf{L}$ consists of three parts: the immediate exit set $\mathbf{N}^-$, the push-forward by φ^0 of the closure of the immediate exit set $\rho(cl(\mathbf{N}^-) \setminus \mathbf{N}^-, \mathbf{N}, \varphi^0)$, and, finaly, the union of the unstable manifolds of $M_{d_i}(p, i)$, $\bigcup_{p=2}^{P_i} W^u_{\mathbf{B}(i)}(M_{d_i}(p, i))$. The union of the first two sets gives $\rho(cl(\mathbf{N}^-), \mathbf{N}, \varphi^0)$ in (8), since $\mathbf{B}(i)$ is an isolating block (see the proof of Lemma 2.2) for a flow on each fiber $\mathbf{R}^{2n} \times \{\lambda\}$, $\lambda \in [c_i, d_i]$, and thus $\rho(\mathbf{N}^-, \mathbf{N}, \varphi^0) = \mathbf{N}^-$. It also follows that

$$\rho(cl(\mathbf{N}^-) \setminus \mathbf{N}^-, \mathbf{N}, \varphi^0) \subset R^{2n} \times \{c_i\}.$$

Given $x \in \mathbf{R}^{2n}$ and $\delta > 0$, let $B_\delta(x) := \{y \in \mathbf{R}^{2n} \mid ||x - y|| < \delta\}$ and given $Y \subset \mathbf{R}^{2n}$, let $B_\delta(Y) := \bigcup_{y \in Y} B_\delta(y)$. Define

$$Q_\delta^- := \bigcup_{i=1}^{I} \bigcup_{p=2}^{P_i} B_\delta(M_{d_i}(p, i)),$$

and let

$$\mathbf{L}_\delta := \mathbf{N}^- \cup \rho((cl(\mathbf{N}^-) \setminus \mathbf{N}^-) \times [c_i, c_i + \delta], \mathbf{N}, \varphi^0) \cup \rho(cl(Q_\delta^-), \mathbf{N}, \varphi^0),$$

see Figure 7.

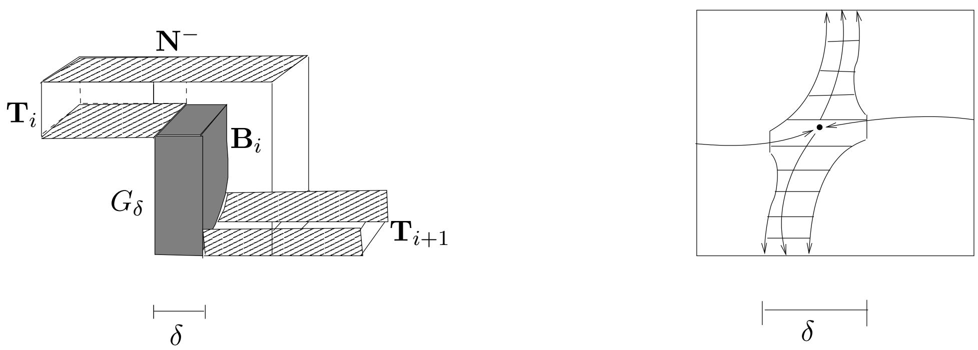

Figure 7: The set $\rho((cl(\mathbf{N}^-) \setminus \mathbf{N}^-) \times [c_i, c_i+\delta], \mathbf{N}, \varphi^0)$ on the left and the push-forward of a neighborhood $B_\delta(M_{d_i}(p,i))$ on the right.

Theorem 3.1 *There exists $\delta_1 > 0$ such that for all $\delta \in (0, \delta_1]$ the diagram below*

$$\begin{array}{ccc} H^*(\mathbf{N}, \mathbf{L}_\delta) & \xrightarrow{\Psi^*_\delta} & H^*(\mathbf{T}_1, \mathbf{T}_1^-) \\ & \overset{\xi^*_\delta}{\searrow} & \downarrow \Theta_\delta \\ & & H^*(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-) \end{array} \tag{11}$$

commutes and Ψ^*_δ is an inclusion induced isomorphism.

Proof. If $\eta < \delta$ then $\{H^*(\mathbf{N}, \mathbf{L}_\delta) \to H^*(\mathbf{N}, \mathbf{L}_\eta)\}$ is an *direct system*, directed by inclusion. By [7, Theorem A.10] there is a natural isomorphism

$$\lim_{\to} H^*(\mathbf{N}, \mathbf{L}_\delta) \equiv H^*(\mathbf{N}, \mathbf{L}).$$

It now follows from [7, Theorem A.4] that there is δ_1 such that

$$H^*(\mathbf{N}, \mathbf{L}_\delta) \equiv H^*(\mathbf{N}, \mathbf{L}) \tag{12}$$

for all $\delta < \delta_1$. The diagram, formed by diagram (11) and the diagram in Proposition 2.4, joined by inclusion maps, commutes, since all maps are natural. By (12) for $\delta < \delta_1$ these inclusion induced maps are isomorphisms. Therefore, for $\delta < \delta_1$, the diagram (11) commutes. $\square$

From now on we fix $\delta \in (0, \delta_1]$ such that $(cl(\mathbf{N}^-) \setminus \mathbf{N}^-) \times [c_i, c_i+\delta] \subset \mathbf{B}(i)$ for all i. Let

$$G^i = G^i(\delta) := \rho((cl(\mathbf{N}^-) \setminus \mathbf{N}^-) \times [c_i, c_i+\delta], \mathbf{N}, \varphi^0)$$

be the push-forward of the a neighborhood of the closure of $\mathbf{N}^-$ in $\mathbf{N}$, see Figure 7. Let U^i_ϵ be the set of all points $x \in \mathbf{N}$ which pass through the set G^i on their way out of $\mathbf{N}$, i.e.

$$U^i_\epsilon := \{x \in \mathbf{N} \mid \text{ if } \varphi^\epsilon(t_0, x) \in \mathbf{N}^- \ \exists \text{ an interval } [t, t_0] \text{ such that } \varphi^\epsilon([t, t_0], x) \subset G^i\}.$$

For an $x \in U^i_\epsilon$ let $t_1(x)$ be the first such t, i.e.

$$t_1(x) := min\{t \mid \varphi^\epsilon([t, t_0], x) \subset G^i\}.$$

Since δ is fixed and hence $G^i(\delta)$ is a fixed set, by continuity for ϵ sufficiently small each set U_ϵ^i is nonempty. Furthermore, clearly $U_\epsilon^i \cap U_\epsilon^j = \emptyset$ for $i \neq j$. Let

$$C_\epsilon^i = \bigcup_{x \in U_\epsilon^i} \varphi^\epsilon(t_1(x), x)$$

be the set where the ϵ-flow through U_ϵ^i enters the set G^i. Let η_i, $0 < \eta_i < \delta$, be an upper λ bound on the set C_ϵ^i, that is a number such that

$$C_\epsilon^i \supset \mathbf{R}^{2n} \times [c_i, c_i + \eta_i].$$

For each $x \in \mathbf{N}$ let $\lambda(x)$ be the λ coordinate of the point x. For each $x \in U_\epsilon^i$ we define

$$\Lambda_i(x) := \begin{cases} \lambda(\varphi^\epsilon(t_1(x), x)) & \text{if } \lambda(\varphi^\epsilon(t_1(x), x)) \in [c_i, c_i + \eta_i] \\ c_i + \eta_i & \text{if } \lambda(\varphi^\epsilon(t_1(x), x)) > c_i + \eta_i \end{cases}.$$

The function $\Lambda_i(x)$ associates to every $x \in U_\epsilon^i$ the λ coordinate of the point $\varphi^\epsilon(t_1(x), x)$ of entry to the set $C_\epsilon^i \cap (\mathbf{R}^{2n} \times [c_i, c_i + \eta_i])$. If the λ coordinate of the entry is larger than $c_i + \eta_i$ then the value of $\Lambda_i(x)$ is $c_i + \eta_i$. Let $U_\epsilon := \bigcup_{i=1}^I U_\epsilon^i$ and let $G := \bigcup_{i=1}^I G^i$.

Definition 3.2 *For fixed $\delta < \delta^*$ define $p^\epsilon : [0, T] \times \mathbf{N} \to \mathbf{N}$, where $T > 2/\epsilon$, by*

$$p^\epsilon(t, x) = \begin{cases} \varphi^\epsilon(t, x) & \text{if } \varphi^\epsilon(t, x) \in \mathbf{N} \\ \varphi^\epsilon(t_0, x) & \text{if there is } t_0 < t \text{ such that } \varphi^\epsilon(t_0, x) \in \partial\mathbf{N}, x \notin U_\epsilon \\ \varphi^\epsilon(\frac{t_0 - t_1}{\eta}(\Lambda_i(x) - c_i) + t_1, x) & x \in U_\epsilon^i \end{cases}.$$

The time $t_0(x)$ is the first time $\varphi^\epsilon(t, x)$ hits the boundary $\partial\mathbf{N}$.

In the third part of the definition we scale the time so that if $\Lambda_i(x) = c_i$ then $p^\epsilon(t, x) = \varphi^\epsilon(t_1, x)$ and if $\Lambda_i(x) = c_i + \eta_i$ then $p^\epsilon(t, x) = \varphi^\epsilon(t_0, x)$. This will prove crucial in the proof of Theorem 3.4.

Theorem 3.3 *There exists $\epsilon^* > 0$ such that for all $\epsilon < \epsilon^*$*

$$p^\epsilon(T, A) \subset \mathbf{L}_\delta \cup \mathbf{T}_1.$$

Proof. Assume that the theorem does not hold. Then there are sequences $x_n \in A$ and $\epsilon_n \to 0$ where

$$(y_n, \lambda_n) := p^{\epsilon_n}(T_n, x_n) \notin \mathbf{L}_\delta \cup \mathbf{T}_1. \tag{13}$$

We note that $p^{\epsilon_n}(T_n, x_n) = \varphi^{\epsilon_n}(t_0(x_n), x_n)$ by definition of maps $p^{\epsilon_n}(T_n, x_n)$. Choosing a subsequence if necessary, let

$$(y, \lambda^*) := \lim_{n\to\infty} (y_n, \lambda_n) = \lim_{n\to\infty} \varphi^{\epsilon_n}(t_0(x_n), x_n).$$

Then $(y, \lambda^*) \notin \operatorname{int}\mathbf{L}_\delta$. We first observe that since $G \subset \mathbf{L}_\delta$ and by (13) we must have $(y_n, \lambda_n) \notin G$. But $(y_n, \lambda_n) = p^{\epsilon_n}(T_n, x_n) \in \partial\mathbf{N} \cup G$ by definition of p^ϵ and thus we conclude $(y_n, \lambda_n) \in \partial\mathbf{N}$. Since $\partial\mathbf{N}$ is closed, we have $(y, \lambda^*) \in \partial\mathbf{N}$.

Assume first $\lambda^* \neq c_i, d_i$ for all i. Since $(y_n, \lambda_n) \in \partial\mathbf{N}$ we have $(y_n, \lambda_n) \in \partial\mathbf{N}_{\lambda_n}$, where $\mathbf{N}_{\lambda_n} = \mathbf{N} \cap (\mathbf{R}^{2n} \times \lambda_n)$. The set $\mathbf{N}_{\lambda^*}$ is an isolating block under the flow ψ^{λ^*} and thus its boundary consists of strict entrance and strict exit set (see [11]). Since $(y, \lambda^*) \notin \text{int } \mathbf{L}_\delta$, and the set $\mathbf{L}_\delta$ contains all strict exit points, this implies that (y, λ^*) must belong to strict entrance set of $\partial\mathbf{N}_{\lambda^*}$. Hence for ϵ sufficiently small, $\varphi^\epsilon(t, y)$ is in the interior of $\mathbf{N}$ for all small t. The same is true for a small neighborhood of y (this is what the adjective "strict" in strict entrance set means [11]). Since y_n lies in this neighborhood for large n and $y_n \in \partial\mathbf{N}$, this contradicts the fact that y_n is a first point where the trajectory starting at x_n touches the boundary of $\mathbf{N}$.

Now we assume that $\lambda^* = c_i$. If $y \in \partial\mathbf{N}_{\lambda^*}$ then the previous argument applies. However, y can be in the interior of the slice $\mathbf{N}_{\lambda^*}$ and still be in the boundary of $\partial\mathbf{N}$. Since $\lambda^* = c_i$ there is an n_0 such that for all $n > n_0$ $\lambda_n \in [c_i, c_i + \eta_i]$. The set G^i is φ^0 invariant and so for sufficiently small ϵ, we have

$$(y_n, \lambda_n) = \varphi^{\epsilon_n}(t_0(x_n), x_n) \in G^i.$$

Since G^i is closed, $(y, \lambda^*) \in G^i \subset \mathbf{L}_\delta$, a contradiction.

Finally, we assume that $\lambda^* = d_i$ for some i. Fix time τ and let

$$(z_\tau, \mu) := \lim_{\epsilon_n \to 0} \varphi^{\epsilon_n}(t_0(x_n) - \tau, x_n). \tag{14}$$

The points on the right hand side of (14) are all in $\mathbf{N}$, $\mathbf{N}$ is compact and so there is a convergent subsequence. Hence the limit is well defined for all τ. Since $t_0(x_n) > 2/\epsilon_n \to \infty$ as $\epsilon_n \to 0$, and τ is fixed, clearly $\mu = \lambda^* = d_i$.

Let $q^\tau_\epsilon := \varphi^\epsilon(t_0(x_n) - \tau, x_n)$. Then clearly

$$\varphi^{\epsilon_n}(\tau, q^\tau_\epsilon) = y_n \in \mathbf{N} \text{ for all } n \text{ and } \tau,$$

and

$$z_\tau := \lim_{n \to \infty} q^\tau_{\epsilon_n}.$$

By continuity

$$\varphi^0(\tau, z_\tau) = y \text{ for all } \tau. \tag{15}$$

If we set $z := \lim_{\tau \to \infty} z_\tau$ then

$$\varphi^0(\tau, z) \in \mathbf{R}^{2n} \times \{d_i\} \qquad \text{for all } \tau$$

and hence z lies in the invariant set under the flow φ^0. It follows from (15) that y has to lie on the unstable manifold of z, which is part of the invariant set of φ^0 in $\mathbf{R}^{2n} \times d_i$.

But such a set is in the interior of $\mathbf{L}_\delta$, so $y \in \text{int}\mathbf{L}_\delta$, a contradiction. □

Theorem 3.4 *There is an $\epsilon^* > 0$ such that for all $\epsilon < \epsilon^*$, the function p^ϵ is continuous.*

Proof. Since the flow is continuous, the only place where we have to check continuity is when the flow exits $\mathbf{N}$ through the boundary. By Theorem 3.3 the place where the flow exits $\mathbf{N}$ is a subset of $\mathbf{L}_\delta \cup \mathbf{T}_1$. If the flow exits $\mathbf{N}$ transversally, then the nearby trajectories will have almost the same exit time and the map is continuous. Hence if

$$(t_0, x) \in \mathbf{N}^- \cup \rho(cl(Q_\delta^-), \mathbf{N}, \varphi^0) \cup \mathbf{T}_1$$

the map is continuous at (x, t). The only place where continuity remains to be checked are $x \in \mathbf{N}$ such that trajectory through x exit through G^i for some i. We have denoted the set of these x by U_ϵ^i.

Take $\epsilon^* > 0$ so small that $U_\epsilon^i \neq \emptyset$ for all i and all $\epsilon < \epsilon^*$. Fix such an ϵ and select a i. Clearly, p^ϵ is continuous in the interior of U_ϵ^i i.e. for those x with $\Lambda_i(x) \in (c_i, c_i + \eta_i)$. We show that p is also continuous for $x \in \partial U_\epsilon^i$. Take $x \in U_\epsilon^i$ with $\Lambda_i(x) = c_i$ for some i. Observe that there is a sequence $\{x_n\} \subset (\mathbf{N} \setminus U_\epsilon^i)$ converging to x such that $\lim_{n\to\infty} t_0(x_n) = t_1(x)$. By the third part of the definition of p^ϵ the function is continuous at x. Similarly one shows that p^ϵ is continuous at $x \in U_\epsilon^i$ with $\Lambda_i(x) = c_i + \eta_i$. Since i was arbitrary, p^ϵ is continuous on U_ϵ.

Notice, that if $U_\epsilon^i = \emptyset$ for some i the function p^ϵ would be discontinuous. $\square$

4 Proof of Theorem 1.4

We need two short lemmas. We assume all assumptions of Theorem 1.4 in this section.

Lemma 4.1 *Let $P^\epsilon := p^\epsilon(T, \cdot) : \mathbf{N} \to \mathbf{L}_\delta \cup \mathbf{T}_1$ and let $e : (\mathbf{T}_1, \mathbf{T}_1^-) \hookrightarrow (\mathbf{L}_\delta \cup \mathbf{T}_1, L_\delta)$ be an inclusion. For $\epsilon < \epsilon^*$ the element*

$$\zeta_\delta^* \circ P^{\epsilon *} \circ (e^*)^{-1}(\beta)$$

is a nontrivial element of $H^l(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-)$.

Proof. By Lemma 2.2 the set B is the support of the generator β of $H^l(\mathbf{T}_1, \mathbf{T}_1^-)$. Observe that e^* is an excision isomorphism and so $(e^*)^{-1}$ is well defined. By Theorem 3.3 the map P^ϵ maps the pair $(\mathbf{N}, \mathbf{L}_\delta)$ to the pair $(\mathbf{L}_\delta \cup \mathbf{T}_1, \mathbf{L}_\delta)$. By Theorem 3.4 the map P^ϵ is homotopic to identity via homotopy p^ϵ. Hence the map

$$P^{\epsilon *} \circ (e^*)^{-1} : H^l(\mathbf{T}_1, \mathbf{T}_1^-) \to H^l(\mathbf{L}_\delta \cup \mathbf{T}_1, \mathbf{L}_\delta) \to H^l(\mathbf{N}, \mathbf{L}_\delta),$$

is an isomorphism. Furthermore, since e^* is induced by inclusion, the isomorphism $P^{\epsilon *} \circ (e^*)^{-1}$ can be identified with the isomorphism Ψ_δ^* from (11). It then follows from the diagram (11) that

$$\zeta_\delta^* \circ P^{\epsilon *} \circ (e^*)^{-1}(\beta) = \Theta_\delta^*(\beta).$$

Since β is a generator and $\Theta_\delta^* \neq 0$ we must have that $\Theta_\delta^*(\beta)$ is a nontrivial element of $H^l(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-)$. $\square$

Corollary 4.2 *Let $\gamma := \zeta_\delta^* \circ P^{\epsilon *} \circ (e^*)^{-1}(\beta) \in H^l(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-)$. Recall that α generates $H_k(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-)$. Then $\gamma \frown \alpha$ is nontrivial in $H_0(\mathbf{T}_{-1})$.*

Proof. Since by Lemma 2.2 $\mathbf{T}_{-1}$ is connected, the group $H_0(\mathbf{T}_{-1})$ is one dimensional. By Lemma 4.1 γ is a nontrivial element of $H^l(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-)$. By Lemma 2.2 the element α generates $H_k(\mathbf{T}_{-1}, \mathbf{T}_{-1}^-)$. Finally, by Lemma 2.5 their pairing is nontrivial in $H_0(\mathbf{T}_{-1})$. $\square$

Proof of Theorem 1.4

Recall that $supp(\alpha) = A \cap \mathbf{T}_{-1}$ and $supp(\beta) = B \cap \mathbf{T}_1$. It follows that non-triviality of the pairing

$$\gamma \frown \alpha \in H_0(\mathbf{T}_{-1})$$

(Corollary 4.2) implies

$$supp(\gamma) \cap supp(\alpha) \neq \emptyset.$$

Take $\mathbf{x} := (u, v) \in A \cap supp(\gamma)$. Since both ζ and e are inclusions, we have

$$P^\epsilon(supp(\gamma)) = P^\epsilon(supp(\zeta_\delta^* \circ P^{\epsilon *} \circ (e^*)^{-1}(\beta)) = supp(\beta) = B.$$

Therefore $P^\epsilon(\mathbf{x}) \in B$. Let

$$\mathbf{y}(t) := p^\epsilon([0, T], \mathbf{x}) = \varphi^\epsilon([0, t_0(\mathbf{x})], \mathbf{x}).$$

Obviously, $\mathbf{y}(t) = (u(t), v(t))$ is a solution of (1) lying in $\mathbf{N}$. Since $\mathbf{y}(0) = \mathbf{x} \in A$ and $\mathbf{y}(t_0) = p^\epsilon(T, \mathbf{x}) = P^\epsilon(\mathbf{x}) \in B$ solution $\mathbf{y}(t)$ satisfies the boundary conditions (2). $\square$

References

[1] Ch. Conley, A qualitative singular perturbation theorem, *Global Theory of Dynamical Systems* (eds. Z.Nitecki and C. Robinson), Lecture Notes in Math., Vol. 819, Springer-Verlag 1980, pp. 65–89.

[2] A. Dold, *Lectures on Algebraic Topology*, Springer-Verlag, Berlin Heidelberg New York, 1972.

[3] T. Gedeon, H. Kokubu, K. Mischaikow, H. Oka and J. Reineck, The Conley index for fast-slow systems I. One dimensional slow variable, *J. of Dynamics and Diff. Eq.* 11, No. 3, (1999), pp. 427-470.

[4] A. Katok and B. Hasselblatt, *Inroduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.

[5] H. Kurland, Following homology in singularly perturbed systems. *J. Differential Equations* 62 (1986), No. 1, 1-72.

[6] H. Kurland, Intersection pairings on Conley indices, *Mem. Amer. Math. Soc.* 119 (1996), No. 571.

[7] W. S. Massey, *Homology and Cohomology Theory* , Marcel Dekker, New York, Basel, 1978.

[8] Ch. McCord, Poincaré-Lefschetz duality for homology Conley index, *Trans. of the Amer. Math. Soc*, 329, No. 1, (1992), pp. 233-252.

[9] C. McCord and K. Mischaikow, Connected simple systems, transition matrices and heteroclinic bifurcations, *Trans. AMS*, **333** (1992), 397–422.

[10] C. McCord and K. Mischaikow, Equivalence of topological and singular transition matrices in the Conley index, *Mich. Math. J.*, **42** (1995), 387–414.

[11] K. Mischaikow. M. Mrozek and J. Reineck, Singular index pairs,*J. Dynam. Differential Equations* 11, No. 3,(1999), 399-425.

[12] D. Salamon, Connected simple systems and the Conley index of isolated invariant sets, *Trans. AMS*, **291** (1985), 1–41.