Upper Semicontinuity of Morse Sets of a Discretization of a Delay-Differential Equation

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In this paper, we consider a discrete delay problem with negative feedback $\dot{x}(t) = f(x(t), x(t-1))$ along with a certain family of time discretizations with stepsize $1/n$. In the original problem, the attractor admits a nice Morse decomposition. We prove that the discretized problems have global attractors. It was proved in [G,M] that such attractors also admit Morse decompositions. We then prove certain continuity results about the individual Morse sets, including that if $f(x, y) = f(y)$, then the individual Morse sets are upper semicontinuous at $n = \infty$.

1. Introduction.

In this paper, we consider the discrete delay problem with negative feedback

$$\begin{aligned} \dot{x}(t) &= f(x(t), x(t-1)) \\ x(t) &= \phi(t), \quad t \in [-1, 0] \end{aligned} \tag{1.1$_\infty$}$$

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and the following time discretization of the problem

$$\begin{aligned}
\dot{y}_0(t) &= f(y_0, y_n)\\
\dot{y}_1(t) &= n(y_0 - y_1)\\
&\vdots\\
\dot{y}_n(t) &= n(y_{n-1} - y_n)\\
\\
y_0(0) &= \phi(0)\\
y_1(0) &= \phi(-\frac{1}{n})\\
&\vdots\\
y_n(0) &= \phi(-1)
\end{aligned} \tag{1.1$_n$}$$

Notice that $\dot{y}_k(0)$ is the slope of the secant line from $\phi(-k/n)$ to $\phi(-(k-1)/n)$.

We consider the case when both problems $(1.1)_\infty$ and $(1.1)_n$ admit a Morse decomposition and we prove certain continuity properties of the individual Morse sets with respect to the discretization parameter n.

Assume that the function f satisfies the following, which we will refer to collectively as assumption (**A1**).

A1a. $f : I\!R^2 \to I\!R$ is C^∞
A1b. $\eta f(0, \eta) < 0$ for all $\eta \neq 0$
A1c. $A + B < 0$ where $A = \partial f(\xi, \eta)/\partial\eta|_{(0,0)}$ and $B = \partial f(\xi, \eta)/\partial\eta|_{(0,0)}$

Notice that the first two conditions also imply that $f(0, 0) = 0$ and $B < 0$.

We will also assume that $(1.1)_\infty$ admits a global attractor. To state this assumption precisely, we must specify the function space in which we usually consider $(1.1)_\infty$ and define the flow in that space. Choose an initial condition $\phi \in C := C([-1, 0], I\!R)$ and let $x(t)$ be the solution with $x(\theta) = \phi(\theta)$ for $\theta \in [-1, 0]$. We can define a solution of $(1.1)_\infty$ as an element in C by defining the function $x_t \in C$ as $x_t(\theta) = x(t + \theta)$ for $\theta \in [-1, 0]$. We then define the solution operator $T_\infty(t)\phi = x_t$. The collection $\{T_\infty(t)\}_{t\geq 0}$ is a semigroup and the action of this semigroup on C defines a semiflow. We denote the set of all bounded solutions of $(1.1)_\infty$ as $\hat{A}_\infty \subset C((-\infty, \infty), I\!R)$ and define $\mathcal{A}_\infty \subset C$ as the set of all initial conditions which give rise to a solution in $\hat{A}_\infty$. The semiflow given by $\{T_\infty(t)\}_{t\geq 0}$ can be extended to a flow $\{T_\infty(t)\}_{t\in\Re}$ on $\mathcal{A}_\infty$. We then assume that

A2. $\mathcal{A}_\infty$ is a global attractor and $\hat{A}_\infty$ admits a Morse decomposition.

Using the results in [H] and assumption **A2**, we will be able to show that $(1.1)_n$ admits a global attractor for large n. It was proved in [G,M] that if there is a global attractor then the global attractor admits a Morse decomposition.

In the remainder of this introduction, we will give the Morse decompositions and state the continuity results.

Since the Morse decomposition will be given in $\hat{A}_\infty$, we must define the flow on $\hat{A}_\infty$. The flow will just be translation by time. If $\hat{x} \in \hat{A}_\infty$, then $\phi := \hat{x}|_{[-1,0]} \in \mathcal{A}_\infty$ and $\hat{x}$ is the solution through initial condition ϕ. For $\theta \in (-\infty, \infty)$, define $\hat{x}_t(\theta) := \hat{x}(t+\theta)$.

In [M-P], the author defined a discrete Lyapunov function on $\hat{A}_\infty$, $V : C((-\infty,\infty), I\!R) \to I\!N$. Define $\sigma := \inf\{t \geq 0 : \hat{x}(t) = 0\}$ if it exists. Then, if σ exists, define $V(\hat{x})$ to be the number of zeroes, counting multiplicity, of $\hat{x}$ in the interval $(\sigma - 1, \sigma]$. Otherwise, define $V(\hat{x}) = 1$. The author then proved that, in $\hat{A}_\infty$, V is bounded above and takes odd integer values and $V(\hat{x}_t)$ is nonincreasing in t.

The Morse sets will be sets in $\hat{A}_\infty$ on which the Lyapunov function is constant. The number of Morse sets will depend on the number of eigenvalues of the linearization which have positive real part. Specifically, the characteristic equation obtained by setting $x = e^{\lambda t}$ in the linearization of $(1.1)_\infty$ is

$$-\lambda + A + Be^{-\lambda} = 0. \tag{1.2}$$

We assume

A3$_\infty$ The zero solution of $(1.1)_\infty$ is hyperbolic; that is there are no roots of 1.2 with zero real part.

Then, if N is the number of roots of 1.2 with positive real part, there are $N^* := N/2 + 1$ Morse sets (it is proved in [M-P] that N is even). These sets are defined as follows. For $1 \leq k \leq N^* - 1$

$$S_k := \{\hat{x} \in \hat{A}_\infty \setminus \{0\} : V(x_t) = 2k - 1 \text{ for all } t \text{ and } 0 \notin \alpha(\hat{x}) \cup \omega(\hat{x})\}$$

and

$$S_{N^*} := \{0\}.$$

It is proved in [M-P] that the sets $\{S_k\}_{1 \leq k \leq N^*}$ form a Morse decomposition of $\hat{A}_\infty$.

To describe the Morse decomposition for $(1.1)_n$, we begin by defining operators analogous to T_∞. If $\mathbf{y_0}$ is an initial condition in $I\!R^{n+1}$ then $\tilde{T}_n(t)\mathbf{y_0} \in I\!R^{n+1}$ will be the solution through $\mathbf{y_0}$ at time t. We will eventually drop the tilde notation when we have chosen a function space in which we can compare solutions of $(1.1)_n$ with solutions of $(1.1)_\infty$.

Systems of the form in $(1.1)_n$ are commonly known as *cyclic feedback systems* and have been studied by Mallet-Paret and Smith [M-P,S], Gedeon and Mischaikow [G,M], and Gedeon[G]. In fact, assumption **A1**b guarantees that this is a *negative cyclic feedback system*. In [M-P,S], Mallet-Paret and Smith define a discrete Lyapunov function for $(1.1)_n$, in the case that it admits a global attractor $\tilde{\mathcal{A}}_n$. For $1 \le i \le n$, define $\delta_i = 1$ and define $\delta_0 = -1$. For a vector $\langle x_0, ... x_n \rangle \in I\!R^{n+1}$ with $x_i \neq 0$, define

$$\tilde{V}_n(\langle x_0, x_1, ..., x_n \rangle) = \text{ card}\{i : \delta_i x_i x_{i-1} < 0\}$$

where we define $x_{-1} = x_n$. $\tilde{V}_n$ counts the number of sign changes in the vector and adds one if the first and last element have the same sign. We extend $\tilde{V}_n$ by continuity whenever possible. If the vector $\mathbf{x}(t) = \langle x_0(t), x_1(t), ..., x_n(t) \rangle$ is a solution of $(1.1)_n$, then $\tilde{V}_n(\mathbf{x}(t))$ is nonincreasing. More precisely, if $\mathbf{x}(t)$ is in a region where $\tilde{V}_n$ is defined, then $\tilde{V}_n(\mathbf{x}(t))$ is constant. If $\tilde{V}_n$ is not defined at $\mathbf{x}(t)$, then for small ϵ, $\tilde{V}_n(\mathbf{x}(t-\epsilon)) = 2 + \tilde{V}_n(\mathbf{x}(t+\epsilon))$ ($\tilde{V}_n(\mathbf{x}(t+\epsilon)) < \tilde{V}_n(\mathbf{x}(t-\epsilon))$) . Clearly, $\tilde{V}_n$ is bounded and takes odd integer values. We will use this Lyapunov function to define the Morse sets. Again the number of Morse sets depends on the number of eigenvalues with positive real part. In section 5 we prove that if $\mathbf{A3}_\infty$ holds then

$\mathbf{A3}_n$ The zero solution of the linearization of $(1.1)_n$ is hyperbolic.

Let K_n be the number of eigenvalues with positive real part. If K_n is even, define $K_n^* := K_n/2+1$; if K_n is odd, define $K_n^* := (K_n+1)/2+1$. There are K_n^* Morse sets. These are defined as follows. For $1 \le k \le K_n^* - 1$,

$$\tilde{S}_k^n := \{\mathbf{x} \in \tilde{\mathcal{A}}_n : \tilde{V}_n(\tilde{T}_n(t)\mathbf{x}) = 2k-1 \text{ for all } t, 0 \notin \alpha(\mathbf{x}) \cup \omega(\mathbf{x})\}$$

and

$$\tilde{S}_{K^*}^n := \{0\} \cup \{\mathbf{x} \in I\!R^{n+1} : \tilde{V}_n(\mathbf{x}) \ge 2K_n^* - 1\}$$

It is proved in [G,M] that this indeed gives a Morse decomposition of $\tilde{\mathcal{A}}_n$.

Hence for each problem $(1.1)_n$, $n \le \infty$, we have an attractor and a Morse decomposition. These Morse decompositions are not unrelated. But it is not clear how we can compare these two problems. It turns out that both the infinite dimensional problem and the finite dimensional are connected to the following distributed delay problem.

$$\begin{aligned} \dot{x}(t) &= f\left(x(t), \int_{-\infty}^0 x(t+s)Q_n(s)ds\right) \\ x_0 &= \phi, \quad \phi \in C((-\infty, 0]) \end{aligned} \tag{1.3$_n$}$$

where

$$Q_n(s) = n^n \frac{(-s)^{n-1}}{(n-1)!} e^{ns}$$

With this kernel, we obtain the system in $(1.1)_n$ if we make the following change of variables (see [B,T])

$$\begin{aligned} y_0(t) &= x(t) \\ y_k(t) &= \int_{-\infty}^0 x(t+s) r_k^n(s) ds \end{aligned}$$

where

$$r_k^n(s) = n^k \frac{(-s)^{k-1}}{(k-1)!} e^{ns}$$

The initial conditions will be

$$\begin{aligned} y_0(0) &= \phi(0) \\ y_k(0) &= \int_{-\infty}^{0} \phi(s) r_k^n(s) ds \end{aligned} \tag{1.4}$$

Problem $(1.1)_\infty$ is the "limit" of the problems $(1.3)_n$ in the sense that the kernels Q_k converge weakly to the δ-function at -1; that is, for bounded functions x (in fact, for functions in the space X defined below), we have

$$\int_{-\infty}^{0} x(s) Q_n(s) ds \to x(-1) \quad \text{as } n \to \infty.$$

The convergence of the kernels allows us to make use of results in [H] about the dependence of attractors on the delay. To state these results, we must first give a function space in which we can compare solutions for different values of n. The choice of function space is discussed extensively in [H]. We choose the space

$$X := \{\phi : (-\infty, 0] \to I\!R | \phi \text{ is continuous on } [-2,0] \text{ and } \|\phi\|_X < \infty\}$$

where

$$\|\phi\|_X := \sup_{-2 \le s \le 0} |\phi(s)| + \int_{-\infty}^{0} |\phi(s)| Q_1(s) ds.$$

It can be shown that the problems $(1.3)_n$ and $(1.1)_\infty$ are well-defined in X. The attractor for $(1.1)_\infty$ in X is just the backward flow through all elements in the attractor in C. This construction is discussed completely in [H]. We will call the attractor in X $\mathcal{A}_\infty$ also. We define solution operators $T_n(t)$ for $(1.3)_n$ in X, $1 \le n \le \infty$ analogous to the operators $\tilde{T}_n(t)$. If the following assumption is satisfied, then for large n, $(1.3)_n$ admits a global attractor in X (otherwise the attractors are local in a large ball whose radius goes to infinity as n does).

A4 There is a fixed bounded set $\Gamma \supset \mathcal{A}_\infty$ into which the orbit $T_n(t)B$ eventually enters and remains for every bounded set $B \subset X$.

Remark 1.0. *We will show that this assumption is satisfied if there is a $u \in I\!R$ such that for $x \in I\!R$ and $y \in I\!R^n$, $xf(x,y) < 0$ if $x \ge u$ and $|y| \le x$.*

The attractors $\mathcal{A}_n$ are upper semicontinuous at $n = \infty$. It is also shown that solutions are continuous with respect to n uniformly for bounded sets of initial conditions and compact intervals of time. We can use these two facts to great advantage to discuss continuity of the Morse sets. First, however, we must give the Morse decomposition for $(1.3)_n$ in terms of $(1.1)_n$. The Morse decomposition in $\tilde{\mathcal{A}}_n$ gives rise to a Morse decomposition in $\mathcal{A}_n$. Define

$$S_k^n := \{\phi \in \mathcal{A}_n | \langle \phi(0), \int_{-\infty}^0 \phi(s) r_1^n(s) ds, \int_{-\infty}^0 \phi(s) r_2^n(s) ds, ..., \int_{-\infty}^0 \phi(s) r_n^n(s) ds \rangle \in \tilde{S}_k^n\}$$

We will prove in section 1 that this indeed gives a Morse decomposition of $\mathcal{A}_n$.

Finally, we now state the results that will be proven in this paper. We begin with the most general result which holds with no further assumptions on $(1.1)_\infty$.

Theorem 1.1. *Assume that assumptions* **A1** *through* **A4** *are satisfied. Then for any* $\epsilon > 0$, *there exists* N *so that for all* $n > N$, *the following hold.*

a. $N^* = K_n^*$; *that is, the number of Morse sets in the decompositin of* $\mathcal{A}_\infty$ *is the same as the number of Morse sets in the decomposition of* $\mathcal{A}_n$.
b. S_k^n *is in an* ϵ*-neighborhood of*

$$M_k := \left(\cup_{j \leq k} S_j^\infty\right) \cup \left(\cup_{j,l \leq k} C^\infty(j,l)\right)$$

for all $1 \leq k \leq N^*$ *where* $C^\infty(j,l)$ *is the set of all connecting orbits with* α*-limit set in* S_j^∞ *and* ω*-limit set in* S_l^∞.

In order to get upper semicontinuity of the individual Morse sets, we require the following extra assumption

A5 Assume that for every solution $x \in \mathcal{A}_\infty$ either $x(t) \to 0$ as $t \to -\infty$ or there are t_1 and t_2 so that between any two zeroes of x in $(-\infty, t_1] \cup [t_2, \infty)$, there is precisely one zero of $\dot{x}$.

It will be shown in section 7 that this assumption holds for equations of the form

$$\dot{x}(t) = f(x(t-1))$$

With this we have the Theorem

Theorem 1.2. *Assume that assumptions* **A1** *through* **A5** *are satisfied. Then for any* $\epsilon > 0$, *there exists* N *so that for all* $n > N$, *the following hold.*

a. $N^* = K_n^*$; *that is, the number of Morse sets in the decompositin of* $\mathcal{A}_\infty$ *is the same as the number of Morse sets in the decomposition of* $\mathcal{A}_n$.
b. S_k^n *is in an* ϵ*-neighborhood of* S_k^∞ *for all* $1 \leq k \leq N^*$

2. The Morse Decomposition of $\mathcal{A}_n$

In the introduction, we gave a Morse decomposition for $\tilde{\mathcal{A}}_n \subset I\!R^{n+1}$ and we defined the sets S_k^n which will make up the decomposition for $\mathcal{A}_n \subset X$. In this definition, we use a natural identification of functions in X to vectors in $I\!R^{n+1}$ via the change of variables. We begin here by defining the bounded linear operator $L_n : X \to I\!R^{n+1}$

$$[L_n\phi](t) := \langle \phi(t), \int_{-\infty}^{0} \phi(s+t) r_1^n(s) ds, ..., \int_{-\infty}^{0} \phi(s+t) r_n^n(s) ds \rangle$$

Lemma 2.1. *L_n is linear for all n and is bounded uniformly in n. Hence it is continuous uniformly in n.*

Proof. The fact that L_n is linear is clear. We will prove that L_n is bounded uniformly in n. Let $\phi \in X$. We look at an individual term in the vector $L\phi$.

$$\begin{aligned} \left|\int_{-\infty}^{0} \phi(s) r_k^n(s) ds\right| &\leq \int_{-\infty}^{-2} |\phi(s)| r_k^n(s) ds + \int_{-2}^{0} |\phi(s)| r_k^n(s) ds \\ &\leq \int_{-\infty}^{-2} |\phi(s)| r_k^n(s) ds + \sup_{-2 \leq s \leq 0} |\phi(s)| \end{aligned}$$

since for any n and k, $\int_{-\infty}^{0} r_k^n(s) ds = 1$. We want to show that this expression is bounded above by $c\|\phi\|_X$ for some c. In this norm, the kernel is $Q_1(s) = e^s$. Define $h_k^n(s) := e^{-s} r_k^n(s)$ so that $r_k^n = h_k^n Q_1$. Notice that h_k^n achieves its maximum value at $s^* = (1-k)/(n-1) \geq -1$ and that h_k^n is increasing for $s < s^*$. Then we have

$$\left|\int_{-\infty}^{0} \phi(s) r_k^n(s) ds\right| \leq h_k^n(-2) \int_{-\infty}^{-2} |\phi(s)| Q_1(s) ds + \sup_{-2 \leq s \leq 0} |\phi(s)|$$

So we must show that $h_k^n(-2)$ is uniformly bounded. If $k = 1$, then $h_k^n(-2) = ne^{-2n}e^2 < 1/2e$. For $k > 1$, we will use Stirling's formula, $m! = m^m e^{-m} \sqrt{2\pi m} e^{\nu/12m}$ for some $0 < \nu < 1$, for $m = k-1$.

$$\begin{aligned} h_k^n(-2) &= \frac{1}{(k-1)!} n^k 2^{k-1} e^{-2n} e^2 \\ &= \frac{e^{k-1}}{(k-1)^{k-1}\sqrt{2\pi(k-1)}} n^k 2^{k-1} e^{-2n} e^2 e^{-\nu/12(k-1)} \\ &\leq \frac{e^{k-1}}{(k-1)^{k-1}\sqrt{2\pi(k-1)}} n^k 2^{k-1} e^{-2n} e^2 \\ &= \left(\frac{n}{k-1}\right)^k \sqrt{\frac{(k-1)}{2\pi}} e^{-n} e^{-(n-k+1)} 2^{k-1} e^2 \end{aligned}$$

Define $j = k - 1$ so that the last line above becomes

$$\begin{aligned} h_k^n(-2) &\le \left(\frac{n}{j}\right)^{j+1} \sqrt{\frac{j}{2\pi}} e^{-n} e^{-(n-j)} 2^j e^2 \\ &= \left(\frac{n}{j}\right)^j e^{-(n-j)} \cdot \frac{n}{j} \sqrt{\frac{j}{2\pi}} e^{-n} 2^j e^2 \\ &= \left(1 + \frac{n-j}{j}\right)^j e^{-(n-j)} \cdot \frac{n}{j} \sqrt{\frac{j}{2\pi}} e^{-n} 2^j e^2 \\ &\le \frac{n}{j} \sqrt{\frac{j}{2\pi}} e^{-n} 2^j e^2 \\ &\le n \sqrt{\frac{n}{2\pi}} \left(\frac{2}{e}\right)^n e^2 \\ &= \frac{1}{n+1} (n+1) n \sqrt{\frac{n}{2\pi}} \left(\frac{2}{e}\right)^n e^2 \\ &\le \frac{3}{2(n+1)} e^2 \end{aligned}$$

where the last estimate is obtained using standard calculus techniques to find the maximum value of the function

$$(x+1)x\sqrt{\frac{x}{2\pi}}\left(\frac{2}{e}\right)^x$$

for $x \ge 0$. This estimate holds for each k and so for each k

$$\begin{aligned} \left|\int_{-\infty}^0 \phi(s) r_k^n(s) ds\right| &\le \frac{3e^2}{2(n+1)} \int_{-\infty}^{-2} |\phi(s)| Q_1(s) ds + \sup_{-2 \le s \le 0} |\phi(s)| \\ &\le \frac{3e^2}{2(n+1)} \|\phi\|_X \end{aligned}$$

Since there are $n+1$ terms in the vector $L\phi$, we have

$$\|L\phi\|_X \le \frac{3e^2}{2} \|\phi\|_X$$

■

What gives us the equivalence between the two systems $(1.1)_n$ and $(1.3)_n$ is the fact that L commutes with the solution operator. In particular

Lemma 2.2. $LT_n(t)\phi = \tilde{T}_n(t)[L\phi]$.

Proof. If $\phi \in X$ and x_t is the solution of $(1.3)_n$ through ϕ and $\langle y_0(t), y_1(t), ..., y_n(t)\rangle$ is the solution of $(1.1)_n$ with initial conditions (1.4), then we have

$$\begin{aligned}\tilde{T}_n(t)[L\phi] &= \tilde{T}_n(t)\Big\langle \phi(0), \int_{-\infty}^{0} \phi(s) r_1^n(s)ds, ..., \int_{-\infty}^{0} \phi(s) r_n^n(s)ds\Big\rangle \\ &= \tilde{T}_n(t)\big\langle y_0(0), y_1(0), ..., y_n(0)\big\rangle \\ &= \big\langle y_0(t), y_1(t), ..., y_n(t)\big\rangle \\ &= \Big\langle x(t), \int_{-\infty}^{0} x(t+s) r_1^n(s)ds, ..., \int_{-\infty}^{0} x(t+s) r_n^n(s)ds\Big\rangle \\ &= LT_n(t)\phi\end{aligned}$$

■

In order to prove that $\{S_k^n\}_{1\le k\le K^*}$ is a Morse decomposition, we will make use of the fact that $\tilde{S}_n^k = L(S_n^k)$. That the sets S_k^n are disjoint, invariant and compact is fairly trivial.

Lemma 2.3. *The sets S_k^n are disjoint, invariant and compact.*

Proof. This is a consequence of Lemma 2.2. Suppose that there is a $\phi \in S_i^n \cap S_j^n$. Then $L\phi \in \tilde{S}_i^n \cap \tilde{S}_j^n$, but since these are disjoint, this cannot be. To prove invariance, suppose $\phi \in S_k^n$. Then

$$LT_n(t)\phi = \Big\langle [T_n(t)\phi](0), \int_{-\infty}^{0} [T_n(t)\phi](s) r_1^n(s)ds, ..., \int_{-\infty}^{0} [T_n(t)\phi](s) r_n^n(s)ds\Big\rangle$$

which is in $\tilde{S}_k^n$ and so $T_n(t)\phi \in S_k^n$. To prove that S_k^n is compact, we prove that is it closed. From assumption **A5** we know that it is bounded. If $\{\phi_i\}$ is a sequence in S_k^n with $\phi_i \to \phi$ in X, then since L is continuous, $L\phi_i \to L\phi$ in $I\!R^{n+1}$. Since $\tilde{S}_k^n$ is closed, $L\phi \in \tilde{S}_k^n$ and so $\phi \in S_k^n$. ■

Next we must show that all solutions approach a set S_k^n.

Lemma 2.4. *For any $\phi \in \mathcal{A}_n$ there are $j \ge i$ so that $\alpha(\phi) \subset S_j^n$ and $\omega(\phi) \subset S_i^n$.*

Proof. First we need to show that $L\phi \in \tilde{\mathcal{A}}_n$. Then we will use the Morse decomposition in $\tilde{\mathcal{A}}_n$. Since $\phi \in \mathcal{A}_n$, it is in $\omega(\mathcal{A}_n)$. Hence there is a sequence of initial conditions $\phi_k \in \mathcal{A}_n$ and a sequence of times t_k so that $T_n(t_k)\phi_k \to \phi$. Now consider the set $\tilde{\Gamma} = \{LT_n(t_k)\phi_k : 1 \le k < \infty\}$. Since L is bounded, the set $\tilde{\Gamma}$ is bounded and so $\omega(\tilde{\Gamma}) \subset \tilde{\mathcal{A}}_n$. But, since L is continuous, $LT(t_k)\phi_k \to L\phi$ and so $L\phi \in \omega(\tilde{\Gamma}) \subset \tilde{\mathcal{A}}_n$.

Since $L\phi \in \tilde{\mathcal{A}}_n$, there exist $j \ge i$ such that $\alpha(L\phi) \subset \tilde{S}_j^n$ and $\omega(L\phi) \subset \tilde{S}_i^n$. To show that $\omega(\phi) \subset S_i^n$, consider $\psi \in \omega(\phi)$. There exists a sequence of times t_k such that $T(t_k)\phi \to \psi$ in X. Then $T(t_k)L\phi = LT(t_k)\phi \to L\psi$, so $L\psi \in \omega(L\phi) \subset \tilde{S}_i^n$. Therefore $\psi \in S_i^n$ and

$\omega(\phi) \subset S_i^n$. The proof that $\alpha(\phi) \subset S_j^n$ follows similarly. In the case that $i = j$, $L\phi \in \tilde{S}_k^n$ and so $\phi \in S_j^n$. ■

We can also define the Lyapunov function in $\mathcal{A}_n$. If $x \in \mathcal{A}_n$, then define

$$V_n(x) = \tilde{V}_n\left(\langle x(0), \int_{-\infty}^{0} x(s)r_1^n(s)ds, ..., \int_{-\infty}^{0} x(s)r_n^n(s)ds\rangle\right) \tag{2.1}$$

V_n has the same properties as $\tilde{V}_n$ and if $x \in S_k^n$ then $V_n(x) = k$.

3. Lower Bounds and Compactness of $\cup_n \mathcal{A}_n$

Besides various convergence properties that we will need, there are three facts which are central to the proof of part b. of Theorem 2.1 and these are presented here.

The first is given as a Theorem in [M-P].

Theorem 3.1a. *Assume that* **A1** *through* **A3** *hold. If $x \in \hat{A}_\infty$ then either $x(t) \to 0$ as $t \to \infty$ or x satisfies the following.*

1. $\liminf_{t\to\infty}(|x(t)| + |\dot{x}(t)|) > C$ *where $C > 0$ is independent of x*
2. *There exist $t_1 > 0$ such that all zeroes of x which lie in $[t_1, \infty)$ are simple.*
3. *There exist $t_2 > 0$ and $d > 0$ such that if z_1 and z_2 are two zeroes of x in $[t_3, \infty)$, then we have $|z_1 - z_2| > d$.*

Since the solution operator is a semigroup in $\mathcal{A}_\infty$, we also have

Corollary 3.1b. *Assume that* **A1** *through* **A3** *hold. If $x \in \hat{A}_\infty$ then either $x(t) \to 0$ as $t \to -\infty$ or x satisfies the following.*

1. $\liminf_{t\to-\infty}(|x(t)| + |\dot{x}(t)|) > C$ *where $C > 0$ is independent of x*
2. *There exist $t_1 < 0$ such that all zeroes of x which lie in $(-\infty, t_1]$ are simple.*
3. *There exist $t_2 < 0$ and $d > 0$ such that if z_1 and z_2 are two zeroes of x in $(-\infty, t_3]$, then we have $|z_1 - z_2| > d$.*

The second fact is that there is a constant $\zeta > 0$ so that if $\mathbf{x} \in S_k^n \subset I\!R^{n+1}$ for any k and n then $\|\mathbf{x}\|_1 > \zeta$ where

$$\|x\|_1 := \max_{0\le i\le n} x_i.$$

This is a result of the following Lemma.

Lemma 3.2. *Let us denote the vector field given by $(1.1)_n$ by $F_n(x)$. There is a constant $\zeta > 0$ such that for all n there is a homeomorphism h_n such that*

$$DF_n(0) \circ h_n(x) = h_n \circ F_n(x)$$

for all $\|\mathbf{x}\|_1 \leq \zeta$.

Lemma 3.2 is standard and follows from the Hartman-Grobman Theorem (see [R]). In the proof, we indicate how to obtain the uniformity.

Proof of Lemma 3.2. Recall that the proof of the Hartman-Grobman Theorem for flows proceeds in three steps. First let $f = Df(0) + g$, $f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a diffeomorphism which satisfies

1. $Lip(g) < \epsilon$ where $Lip(g)$ is the global Lipschitz constant
2. $\epsilon(1-a)^{-1} < 1$ where a is such that spectrum $Df(0) \subset \{\lambda : |\lambda| < a \text{ or } |\lambda^{-1}| < a\}$.

Then there is a C^0-conjugacy h such that $Df(0) \circ h(x) = h \circ f(x)$ for all $x \in \mathbb{R}^{n+1}$.

The second step is the local version of the first step which says that since $g(x) = o(x)$ in the neighborhood of the origin, given "the gap" in the spectrum at a, we can always find a small neighborhood of the origin where $Lip(g) < (1-a)$. Then we modify the function g outside of this neighborhood to obtain the function $\tilde{g}$ with $Lip(\tilde{g}) < (1-a)$ for all $x \in \mathbb{R}^{n+1}$. Then we can use the first part to get a global conjugacy using the function $\tilde{g}$ which becomes a conjugacy with g on the small neighborhood of the origin where $Lip(g) < (1-a)$.

The third step involves taking a time one map of the flow, which is a diffeomorpism, and then applying the above construction and showing that when the time one map is conjugate to a time one map of the linear flow then the conjugacy extends to the time t map for any t, and hence to the entire flow.

The important point for our purposes is that, given a, the conjugacy between the two flows is valid on a neighborhood U where $Lip(g) < (1-a)$.

We have to consider a family of flows, $F_n(x) = DF_n(x) + g_n(x)$. In order to prove the Lemma we need to show that a and U can be chosen independently of n, for large n.

The fact that a can be chosen independently of n is the consequence of Lemma 5.1 and assumption **A3**. For $n \leq \infty$, we denote by $\Lambda(n)$ the spectrum of $(1.1)_n$. Let $\eta > 0$ be such that $\Lambda(\infty) \cap \{z : |\text{Re}(z)| < 2\eta\} = \emptyset$. Then by Lemma 5.1 there is an N such that for all $n > N$ we have that

$$\Lambda(n) \cap \{z : |\text{Re}(z)| < \eta\} = \emptyset.$$

We observe that $a := e^{-\eta}$ is the desired gap.

Now we discuss the Lipschitz constant and the neighborhood. From $(1.1)_n$ we know

$$g_n(\mathbf{x}) = \begin{pmatrix} f(x_0, x_n) - \frac{\partial f}{\partial x_0} x_0 - \frac{\partial f}{\partial x_n} x_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

since $(1.1)_n$ is linear except in the first equation. This function does not change with n. Therefore, there is a constant C such that if $\sqrt{x_0^2 + x_n^2} < C$ then the Lipschitz constant

$Lip(g_n) < (1-a)$. The constant C does not depend on n. If we choose $\zeta = C/2$ then we have that for $\|\mathbf{x}\|_1 < \zeta$, $Lip(g_n) < 1-a$ for all n. This proves the Lemma. ■

Corollary 3.3. *If a solution x^n of $(1.3)_n$ satisfies $|x^n(t)| < \zeta$ for all $t > 0$ then $x^n \in W^s(0)$.*

The final fact is that $\cup_n \mathcal{A}_n$ is compact in X, provided it is bounded in X. Assumption **A4** ensures that it is bounded. In [H], it is proved that a set which is equicontinuous and equibounded in $C((-\infty,0],I\!R)$ is compact in X. The proof is not hard and relies on two facts; first, in such a set in X, a sequence has a subsequence which converges uniformly in compact intervals and second, the weight in the X-norm decays exponentially at $-\infty$. We will use this result here.

Proposition 3.4. *If $\cup_n \mathcal{A}_n$ is bounded in X then it is equicontinuous and equibounded in $C((-\infty,0],I\!R)$.*

Proof. By assumption there is a $K > 0$ so that for every $\phi \in \cup_n \mathcal{A}_n$, we have $\|\phi\|_X < K$. Suppose $\phi \in \mathcal{A}_n$ for some n . We will show that, independent of n, for every $s \in (-\infty, 0]$, $|\phi(s)| < K$. Hence $\sup_{-\infty < s \le 0} |\phi(s)| < K$.

If $-2 \le s \le 0$ then clearly

$$|\phi(s)| \le \sup_{-2 \le s \le 0} |\phi(s)| \le \|\phi\|_X$$

If $s < -2$, then we use the fact that the flow in $\mathcal{A}_n$ under T_n is translation to the left and the fact that $\mathcal{A}_n$ is invariant under T_n. So

$$|\phi(s)| = |[T_n(s)\phi](0)| \le \sup_{-2 \le s \le 0} |T_n(s)\phi| \le \|T_n(s)\phi\|_X \le K$$

Hence if $\cup_n \mathcal{A}_n$ is bounded uniformly in X it is bounded uniformly in $C((-\infty,0])$.

To show that $\cup_n \mathcal{A}_n$ is equicontinuous, we will show that there is an $M > 0$ so that for all $\phi \in \cup_n \mathcal{A}_n$, we have $|\dot{\phi}(s)| < M$ for all $s \in (-\infty, 0]$. Suppose $\phi \in \mathcal{A}_n$ for some n. Since, as above, $\phi(s) = [T_n(s)\phi](0)$ and $\mathcal{A}_n$ is invariant, we really only need to show that for all $\phi \in \cup_n \mathcal{A}_n$, $|\dot{\phi}(0)| < M$. Since

$$|\dot{\phi}(0)| = |f(\phi(0), \int_{-\infty}^{0} \phi(s) Q_n(s) ds)|$$

and f is continuous, we just need to show that $|\phi(0)|$ and $\int_{-\infty}^{0} |\phi(s) Q_n(s)| ds$ are bounded. That $|\phi(0)| < K$ is clear since $|\phi(0)| \le \|\phi\|_X$. So we prove $\int_{-\infty}^{0} |\phi(s)| Q_n(s) ds$ is bounded.

$$\begin{aligned}
\int_{-\infty}^{0} |\phi(s)| Q_n(s) ds &= \int_{-2}^{0} |\phi(s)| Q_n(s) ds + \int_{-\infty}^{-2} |\phi(s)| Q_n(s) ds \\
&\le \sup_{-2 \le s \le 0} |\phi(s)| + \int_{-\infty}^{-2} |\phi(s)| n^n \frac{(-s)^{n-1}}{(n-1)!} e^{(n-1)s} Q_1(s) ds \\
&= \sup_{-2 \le s \le 0} |\phi(s)| + \int_{-\infty}^{-2} |\phi(s)| h_{n-1}^n(s) Q_1(s) ds
\end{aligned}$$

where h_k^n is as in the proof of Lemma 2.1. Since h_{n-1}^n achieves its maximum value at $s^* := -(n-2)/(n-1) > -1$ and h_{n-1}^n is increasing for $s < s^*$, we have as in Lemma 2.1

$$\int_{-\infty}^{0} |\phi(s)| Q_n(s) ds \leq \sup_{-2 \leq s \leq 0} |\phi(s)| + h_{n-1}^n(-2) \int_{-\infty}^{-2} |\phi(s)| Q_1(s) ds$$

Following the remainder of the proof of Lemma 2.1 in which we estimate h_k^n, we get

$$\int_{-\infty}^{0} |\phi(s)| Q_n(s) ds \leq \frac{3e^2}{2} \|\phi\|_X \leq \frac{3e^2 K}{2}$$

where the estimate is independent of n. Hence the derivative is bounded and $\cup_n \mathcal{A}_n$ is equicontinuous. ∎

4. Convergence Results

We will need two basic types of convergence results. First, if a sequence $\{x^n\}$ with $x^n \in \mathcal{A}_n$ converges in X, then it converges uniformly for compact intervals of time and the same holds for derivatives of x^n. Second, the integral

$$\int_{-\infty}^{0} x^n(s+t) r_k^n(s) ds \tag{4.1}$$

is close to

$$x\left(t - \frac{k-1}{n}\right) \tag{4.2}$$

and the analogous fact holds for derivatives of x^n and x. This is because r_k^n has its maximum value at $-(k-1)/n$ and as n gets bigger, each r_k^n looks more and more like a δ-function at $-(k-1)/n$.

We begin with the convergence of sequences x^n.

Lemma 4.1. *Suppose $x^n \in \mathcal{A}_n$ and $x^n \to x \in \mathcal{A}_\infty$ where the convergence is in the X-norm. Then, given ϵ and T, there exists $N(\epsilon, T)$ so that for $n > N$, $|x^n(t) - x(t)| < \epsilon$ for all $t \in [-T, 0]$*

Proof. Remember that functions in $\cup_n \mathcal{A}_n$ are continuous by virtue of the norm and the facts that the flow T_n is translation in $\mathcal{A}_n$ and $\mathcal{A}_n$ is invariant. Since Q_1 is increasing, we have

$$\begin{aligned} Q_1(-T) \int_{-T}^{0} |x^n(s) - x(s)| ds &\leq \int_{-T}^{0} |x^n(s) - x(s)| Q_1(s) ds \\ &\leq \int_{-\infty}^{0} |x^n(s) - x(s)| Q_1(s) ds \\ &\to 0 \end{aligned}$$

Hence $x^n \to x$ for almost every $t \in [-T, 0]$, but since x^n and x are continuous, the convergence is uniform. ■

Before we prove convergence of the derivatives, we must show that the derivatives are bounded

Lemma 4.2. *For any $j \geq 0$, the set $\{\frac{d^j}{dt^j}x : x \in \cup_n \mathcal{A}_n\}$ is bounded and equicontinuous in $C((-\infty, 0], \Re)$.*

Proof. If $x^n \in \mathcal{A}_n$, we use the differential equation $(1.3)_n$ to compute the higher order derivatives. We find that these include the partial derivatives of f up to order j evaluated at

$$\left(x(t), \int_{-\infty}^{0} x(t+s)Q_n(s)ds\right)$$

and powers of $\dot{x}^n$ and $\int_{-\infty}^{0} \dot{x}^n(s)Q_n(s)ds$. In proposition 3.4, we proved that there is an M so that $\|\dot{x}\|_\infty \leq M$ for all $x \in \cup_n \mathcal{A}_n$. By using the same proof that we used in Proposition 3.4 to prove that $\int_{-\infty}^{0} |x(s)|Q_n(s)ds$ is uniformly bounded for $x \in \cup_n \mathcal{A}_n$, we can prove that $\int_{-\infty}^{0} |\dot{x}(s)|Q_n(s)ds$ is uniformly bounded for $x \in \cup_n \mathcal{A}_n$. Then since f is C^∞, we are done. ■

To prove convergence of the derivatives $\frac{d^j}{dt^j}x^n \to \frac{d^j}{dt^j}x$ uniformly as $n \to \infty$, we must prove, in the following order, that

1. $\int_{-\infty}^{0} x^n(t+s)Q_n(s)ds \to x(t-1)$
2. $\dot{x}^n \to \dot{x}$
3. $\int_{-\infty}^{0} \dot{x}^n(t+s)Q_n(s)ds \to \dot{x}(t-1)$.

To this end, we prove the following lemmas.

Lemma 4.3. *Suppose $\{v_n\}$ is a sequence in X and that there is a $v \in X$ so that $v_n(t) \to v(t)$ for all $t \in [-T-2, 0]$ and the sequence $\{v_n\}_0^\infty$ is equicontinuous and equibounded in $C((-\infty, 0], I\!R)$. Also suppose that for each n, $0 \leq k(n) \leq n$ is such that the sequence $\{k(n)/n\}$ converges to some limit l. Then for all ϵ, there exists $N(\epsilon, T)$ so that for all $n > N$,*

$$\left| v\left(t - \frac{k(n)-1}{n}\right) - \int_{-\infty}^{0} v_n(s+t) r_{k(n)}^n ds \right| < \epsilon$$

and

$$\left| v(t-l) - \int_{-\infty}^{0} v_n(s+t) r_{k(n)}^n(s) ds \right| < 2\epsilon$$

for $t \in [-T, 0]$.

Remark 4.4. *Notice that Lemma 4.3 also implies that for any ϵ there is an N so that for all $n > N$*

$$\left| v\left(t - \frac{k-1}{n}\right) - \int_{-\infty}^{0} v_n(s+t) r_k^n(s) ds \right| < \epsilon$$

for all $0 \le k \le n$. One simply applies the first estimate n times, each for some fixed value of k. We will often use the convergence in this form as well as in the form given in the Lemma.

Proof of Lemma 4.3. Recall that $\int_{-\infty}^0 r_k^n(s)ds = 1$ for any k and n. Then

$$\begin{aligned}
&\left| v\left(t - \frac{k(n)-1}{n}\right) - \int_{-\infty}^0 v_n(t+s) r_{k(n)}^n(s)ds \right| \\
&\le \int_{-\infty}^0 \left| v\left(t - \frac{k(n)-1}{n}\right) - v(t+s)\right| r_{k(n)}^n(s)ds + \int_{-\infty}^0 |v(t+s) - v_n(t+s)|\, r_{k(n)}^n(s)ds \\
&:= I_1 + I_2
\end{aligned}$$

First we show that $I_2 \to 0$. We assumed that $\{v_n\}$ is equibounded. Let K be the bound. Then

$$\begin{aligned}
I_2 &= \int_{-\infty}^{-2} |v_n(t+s) - v(t+s)| r_{k(n)}^n(s)ds + \int_{-2}^0 |v_n(t+s) - v(t+s)| r_{k(n)}^n(s)ds \\
&\le 2K \int_{-\infty}^{-2} r_{k(n)}^n(s)ds + \sup_{t-2 \le s \le t} |v_n(s) - v(s)| \\
&\le 2K r_{k(n)}^n(-2) + \sup_{-T-2 \le s \le 0} |v_n(s) - v(s)|
\end{aligned}$$

Convergence of the first term is shown in the proof of Lemma 2.1, where we obtained the inequality

$$r_k^n(-2) = e^{-2} h_k^n(-2) \le n\sqrt{\frac{n}{2\pi}} \left(\frac{2}{e}\right)^n$$

Convergence of the second term is by assumption. Hence there exists N_1 so that for all $n > N_1$, $I_2 < \epsilon/2$.

Next, we show that $I_1 \to 0$. To shorten the notation, we will write k instead of $k(n)$. Let $\delta_1 = n^{-1/3} + n^{-1}$ and $\delta_2 = n^{-1/3} - n^{-1}$. We have

$$\begin{aligned}
I_1 &= \int_{-\infty}^0 \left| v\left(t - \frac{k-1}{n}\right) - v(t+s)\right| r_k^n(s)ds \\
&\le \int_{-\infty}^{-(k-1)/n - \delta_1} \left| v\left(t - \frac{k-1}{n}\right) - v(t+s)\right| r_k^n(s)ds \\
&+ \int_{-(k-1)/n-\delta_1}^{-(k-1)/n + \delta_2} \left| v\left(t - \frac{k-1}{n}\right) - v(t+s)\right| r_k^n(s)ds \\
&+ \int_{-(k-1)/n+\delta_2}^{0} \left| v\left(t - \frac{k-1}{n}\right) - v(t+s)\right| r_k^n(s)ds \\
&:= J_1 + J_2 + J_3
\end{aligned}$$

where we define $r_k^n(s) = 0$ for $s > 0$. For every ϵ, there is an N_2 so that for all $n > N_2$, $J_2 < \epsilon/4$ by the equicontinuity in assumption.

Let K be the bound for $\{v_n\}$. Then for both J_1 and J_3 we have

$$J_1 \leq 2K \int_{-\infty}^{-(k-1)/n-\delta_1} r_k^n(s)ds$$
$$J_3 \leq 2K \int_{-(k-1)/n+\delta_2}^{0} r_k^n(s)ds$$

To estimate these, we will think of r_k^n as a probability distribution and use Chebyshev's inequality.

$$\int_{-\infty}^{\mu_k^n-\nu} r_k^n(s)ds + \int_{\mu_k^n+\nu}^{0} r_k^n(s)ds \leq \frac{(\sigma_k^n)^2}{\nu^2}$$

where μ_k^n is the mean of r_k^n and $(\sigma_k^n)^2$ the variance. We will prove in following Lemmas that $\mu_k^n = -k/n$ and $(\sigma_k^n)^2 = k/n^2$. Then we have

$$\begin{aligned} &\int_{-\infty}^{-(k-1)/n-\delta_1} r_k^n(s)ds + \int_{-(k-1)/n+\delta_2}^{0} r_k^n(s)ds \\ &= \int_{-\infty}^{-k/n-1/\sqrt[3]{n}} r_k^n(s)ds + \int_{-k/n+1/\sqrt[3]{n}}^{0} r_k^n(s)ds \\ &\leq \frac{k\sqrt[3]{n}^2}{n^2} \leq \frac{n\sqrt[3]{n}^2}{n^2} = \frac{1}{\sqrt[3]{n}} \end{aligned}$$

And so

$$J_1 + J_3 \leq \frac{2K}{\sqrt[3]{n}}$$

Hence there is an N_3 so that for all $n > N_3$, $J_1 + J_3 < \epsilon/4$. If we choose $N_4 = \max(N_2, N_3)$ then for all $n > N_4$, $I_1 < \epsilon/2$. Hence if $n > \max(N_1, N_4)$, then $I_1 + I_2 < \epsilon$ and we obtain the first inequality of Lemma 4.3.

¿From this and the fact that v is continuous, the second inequality of Lemma 4.3 is simple since

$$\begin{aligned} \left| v(t-l) - \int_{-\infty}^{0} v_n(t+s)r_k^n(s)ds \right| &\leq \left| v(t-l) - v(t - \frac{k-1}{n}) \right| \\ &+ \left| v(t - \frac{k-1}{n}) - \int_{-\infty}^{0} v_n(t+s)r_k^n(s)ds \right| \end{aligned}$$

■

Finally we prove that $\frac{d^j}{dt^j}x^n(t) \to \frac{d^j}{dt^j}x(t)$. Though this holds for all j provided f is C^∞, we only use it for $j = 1, 2$ and hence we only prove it for those cases here. Higher order derivatives follow in a similar manner.

Lemma 4.5. *Suppose $x^n \in \mathcal{A}_n$ and $x^n \to x \in \mathcal{A}_\infty$ in X. Then, given ϵ and T, there exists $N(\epsilon, T)$ so that for $n > N$, $|\dot{x}^n - \dot{x}(t)| < \epsilon$ for all $t \in [-T, 0]$.*

Proof. By Proposition 3.4 and Lemma 4.1, the sequence x_n satisfies the hypotheses of Lemma 4.3. In that Lemma, choose $k(n) = n$ for all n. Then we have for all $t \in [-T, 0]$,

$$\left| x(t-1) - \int_{-\infty}^{0} x_n(t+s)Q_n(s)ds \right| < \epsilon$$

Since f is C^∞, we also have that, for n large enough,

$$\left| f(x(t), x(t-1)) - f\left(x_n(t), \int_{-\infty}^{0} x_n(t+s)Q_n(s)ds\right) \right| < \epsilon$$

and so we are done ■

So statement 3. before Lemma 4.3 also holds by applying first Lemma 4.5 and them Lemma 4.3 with $v_n = \dot{x}^n$. We have proved the following proposition.

Proposition 4.6. *Let $j \geq 0$. Suppose $x^n \in \mathcal{A}_n$ and $x^n \to x \in \mathcal{A}_\infty$ in X. Then, given ϵ and T, there exists $N(\epsilon, T)$ so that for $n > N$,*

$$\left| \frac{d^j}{dt^j}x^n(t) - \frac{d^j}{dt^j}x(t) \right| < \epsilon$$

for all $t \in [-T, 0]$.

Now we do the computations for the mean and variance.

Lemma 4.7. $\mu_k^n = -k/n$

Proof. We will use induction on k for the proof. For $k = 1$ we have

$$\mu_1^n = \int_{-\infty}^{0} s r_1^n(s)ds = \int_{-\infty}^{0} sne^{ns}ds = -\frac{1}{n}$$

Now suppose that

$$\mu_k^n = \int_{-\infty}^{0} s r_k^n(s)ds = \int_{-\infty}^{0} s\frac{n^k}{(k-1)!}(-s)^{k-1}e^{ns}ds = \frac{-k}{n}$$

Then

$$\begin{aligned}
\mu_{k+1}^n = \int_{-\infty}^0 s r_{k+1}^n(s)ds &= \int_{-\infty}^0 s\frac{n^{k+1}}{k!}(-s)^k e^{ns}ds \\
&= \frac{n^{k+1}}{k!}(-1)^k \int_{-\infty}^0 s^{k+1}e^{ns}ds \\
&= \frac{n^{k+1}}{k!}(-1)^k \left(s^{k+1}\frac{e^{ns}}{n}|_{-\infty}^0 - \int_{-\infty}^0 (k+1)s^k\frac{e^{ns}}{n}ds \right) \\
&= \frac{n^{k+1}}{k!}(-1)^{k+1}\int_{-\infty}^0 (k+1)s^k\frac{e^{ns}}{n}ds \\
&= \frac{k+1}{k}\int_{-\infty}^0 s\frac{n^k}{(k-1)!}(-s)^{k-1}e^{ns}ds \\
&= \frac{k+1}{k}\mu_k^n \\
&= -\frac{k+1}{n}
\end{aligned}$$

■

Lemma 4.8. $(\sigma_k^n)^2 = k/n^2$

Proof. We will use the identity

$$(\sigma_k^n)^2 = \int_{-\infty}^0 s^2 r_k^n(s)ds - (\mu_k^n)^2$$

Evaluating the first term requires integrating by parts.

$$\begin{aligned}
\int_{-\infty}^0 s^2 r_k^n(s)ds &= \int_{-\infty}^0 s^2\frac{n^k}{(k-1)!}(-s)^{k-1}e^{ns}ds \\
&= \int_{-\infty}^0 \frac{n^k}{(k-1)!}(-s)^{k+1}e^{ns}ds \\
&= \int_{-\infty}^0 \frac{n^k}{(k-1)!}\left((-s)^{k+1}\frac{e^{ns}}{n}|_{-\infty}^0 + \int_{-\infty}^0 (k+1)(-s)^k\frac{e^{ns}}{n}ds \right) \\
&= -\frac{k+1}{n}\int_{-\infty}^0 s\frac{n^k}{(k-1)!}(-s)^{k-1}e^{ns}ds \\
&= -\left(\frac{k+1}{n}\right)\mu_k^n \\
&= \left(\frac{k+1}{n}\right)\left(\frac{k}{n}\right) \\
&= \frac{k^2+k}{n^2}
\end{aligned}$$

So

$$(\sigma_k^n)^2 = \frac{k^2+k}{n^2} - \frac{k^2}{n^2} = \frac{k}{n^2}$$

■

5. The Proof of Theorem 1.1a In this section, we will show that the roots of the characteristic equation associated with $(1.1)_n$ converge to the roots of the characteristic equation associated with $(1.1)_\infty$ uniformly on bounded regions of $\mathbb{C}$. With this, we can show that for large n, the number of eigenvalues with positive real part, and hence the dimension of the unstable manifold, is constant. This proves a. of Theorems 1.1 and 1.2.

The characteristic equation associated with $(1.1)_\infty$ is

$$\Delta(\lambda) = (A-\lambda)e^{\lambda} + B = 0 \tag*{$(5.1)_\infty$}$$

The solutions of $(5.1)_\infty$ are isolated and there are only finitely many in the right half plane. We will denote them by λ_j where $\mathrm{Re}(\lambda_j) \geq \mathrm{Re}(\lambda_k)$ for $j<k$.

For the ODE $(1.1)_n$, the linearization about the origin is

$$\begin{aligned} \dot{x}_0 &= Ax_0 + Bx_n \\ \dot{x}_1 &= n(x_0 - x_1) \\ &\vdots \\ \dot{x}_n &= n(x_{n-1} - x_n). \end{aligned}$$

If J_n is the matrix

$$J_n = \begin{pmatrix} A & 0 & 0 & \dots & 0 & B \\ n & -n & 0 & \dots & 0 & 0 \\ 0 & n & -n & \dots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 & 0 \\ \\ 0 & 0 & 0 & \dots & n & -n \end{pmatrix}$$

then the characteristic equation associated with $(1.1)_n$ is

$$\det(J_n - \mu I) = (A-\mu)(-n-\mu)^n + (-n)^n B = 0$$

or, if we divide through by $(-n)^n$,

$$\Delta_n := (A-\mu)\left(1+\frac{\mu}{n}\right)^n + B = 0 \tag*{$(5.1)_n$}$$

We will use μ_i^n to denote roots of $(5.1)_n$ with $\mathrm{Re}(\mu_j^n) \geq \mathrm{Re}(\mu_k^n)$ when $j<k$. $\Delta_n \to \Delta$ uniformly on bounded sets of $\mathbb{C}$.

Lemma 5.1. *There is an $N > 0$ so that for all $n > N$, the number of eigenvalues of $(5.1)_n$ with positive real part is constant and equal to the number of eigenvalues of $(5.1)_\infty$ with positive real part.*

Proof. Define

$$f(\lambda) := \Delta(\lambda)$$
$$g_n(\lambda) := -\Delta(\lambda) + \Delta_n(\lambda) = -(A-\lambda)e^{\lambda} + (A-\lambda)\left(1+\frac{\lambda}{n}\right)^n$$

so $\Delta_n = \Delta + g_n$. Let Γ be any closed curve which contains $\lambda_1, ..., \lambda_m$ but contains no other roots of Δ. Let $\delta = \min_{z\in\Gamma} |f(z)| > 0$. Then there exists N (which must be larger than m), so that for all $z \in \Gamma$, $|g_n(z)| < \delta$ for all $n > N$. Hence, by Rouche's Theorem, $\Delta := f$ and $\Delta_n := f + g_n$ have exactly the same number of roots inside Γ for all $n > N$. If we pick Γ to be in the right half plane so that it contains all the eigenvalues of $(5.1)_\infty$ in the right half plane, then we are done. ■

Lemma 5.2. *For any ϵ and m, there exists N such that for all $1 \le j \le m$ and for all $n > N$,*

$$|\lambda_j - \mu_j^n| < \epsilon$$

Proof. Let f and g_n be as above. Let B_j be the boundary of the ϵ-ball around λ_j. and let $\delta = \min_{z\in B_j} |f(z)| > 0$. We can choose N large enough so that for all $z \in \cup_j B_j$, $|g_n(z)| < \delta$ for all $n > N$. Then, on each B_j we have $|f| > |g_z|$, so inside each B_j we have that both $\Delta_n := f + g_n$ and $\Delta := f$ have exactly one root. Again, this holds for all $n > N$. ■

Remark 5.3. *As a corollary to Lemmas 5.1 and 5.2 we have the assumption* $\mathbf{A3}_n$.

6. The Proof of Theorem 1.1b

For a fixed k, we consider a sequence $x^n \in S_k^n$. This sequence has a convergent subsequence $x^{n_j} \to x$.

Lemma 6.1. *Suppose $x^{n_j} \in S_k^{n_j} \subset \mathcal{A}_{n_j}$ and $x^{n_j} \to x$ in X. Then $x \in \mathcal{A}_\infty$.*

Proof. Suppose $x \notin \mathcal{A}_\infty$ and let $d = \delta(x, \mathcal{A}_\infty) := \inf_{\phi\in\mathcal{A}_\infty} \|x - \phi\|_X$, the distance between x and $\mathcal{A}_\infty$. Choose $\epsilon < d/2$. Since the attractors $\mathcal{A}_n$ are upper semicontinuous, there exists N_1 such that for all $n_j > N_1$, $\mathcal{A}_{n_j} \subset N_\epsilon(\mathcal{A}_\infty)$ so $\delta(x^{n_j}, \mathcal{A}_\infty) < \epsilon$ and there exists N_2 so that for all $n_j > N_2$, $\|x^{n_j} - x\|_X < \epsilon$. Thus for $n > \max(N_1, N_2)$ we have

$$\delta(x, \mathcal{A}_\infty) \le \|x - x^{n_j}\|_X + \delta(x^{n_j}, \mathcal{A}_\infty) < d$$

and so we have reached a contradiction. ■

Lemma 6.2. *Suppose $x^{n_j} \in S_k^{n_j} \subset \mathcal{A}_{n_j}$ and $x^{n_j} \to x$ in X. Then $x(t)$ is a solution of $(1.1)_\infty$ for all t.*

Proof. By the Lemmas in section 4, $x^{n_j} \to x$ uniformly in $[-1,0]$ and also in $[-1,0]$, $\dot{x}^{n_j} \to \dot{x}$ and $\int_{-\infty}^{0} x^{n_j}(t+s)Q_n(s)ds \to x(t-1)$. Therefore $\dot{x}(t) = f(x(t), x(t-1))$ and so $x(s)$ satisfies $(1.1)_\infty$ for $s \in [-1,0]$. Let x_0 be the restriction of x to the interval $[-1,0]$. Since in the attractor the flow associated with $(1.1)_\infty$ is just translation, $x(t) = [T(t)x_0](0)$ and so $x(t)$ satisfies $(1.1)_\infty$ for all t. ■

The goal of this section is to prove that, in fact,

$$x \in M_k = \left(\cup_{j\le k} S_j^\infty\right) \cup \left(\cup_{i,j\le k} C(S_i^\infty, S_j^\infty)\right)$$

Then with the following Lemma, we can complete the proof.

Lemma 6.3. *Suppose that every sequence $\{x^n\}$ with $x^n \in S_k^n$ has a convergent subsequence which converges to an element $x \in M_k$. Then for every ϵ there is an N such that for all $n > N$, $S_k^n \subset N_\epsilon(M_k)$.*

Proof. Suppose this is not true. Then there exist an ϵ so that for all N there is an $\bar{n} > N$ with $S_k^{\bar{n}} \not\subset N_\epsilon(M_k)$. Consider a sequence $N_j \to \infty$ and a corresponding sequence $\bar{n}_j \to \infty$. Then we can construct a sequence $x^{\bar{n}_j}$ so that $x^{\bar{n}_j} \in S_k^{\bar{n}_j}$ but $x^{\bar{n}_j} \notin N_\epsilon(M_k)$. According to the discusion above, this sequence has a convergent subsequence which converges to an element $x \in M_k$, so in fact, for $\bar{n}_j$ large enough, $x^{\bar{n}_j} \in N_\epsilon(M_k)$ and we have reached a contradiction. ■

So for the remainder of this section x will be the limit of a subsequence of $\{x^n\}$ and we will write $\{x^n\}$ for the subsequence as well. In order to use the results of Corollary 3.1b, we must know that $x(t) \not\to 0$ as $t \to -\infty$. For now we state this as Assumption 6.4, however we will prove in Lemma 6.7 that this assumption is indeed satisfied.

Assumption 6.4. *$x(t) \not\to 0$ as $t \to -\infty$.*

Hence, by Corollary 3.1b there exists $t_1 < 0$ such that all zeroes of x in $(-\infty, t_1]$ are simple. Also 2 and 3 of Theorem 3.1 hold. Since $V(x_t)$ takes integer values and is bounded below (by 1) and above, there is a t_2 so that $V(x(t))$ is constant for $t < t_2$. With d and C as in Theorem 3.1, define $\bar{\eta}$ so that

$$\bar{\eta} := \min\left(\frac{2dC}{d+4}, \frac{C}{2}\right)$$

The following Lemma holds

Lemma 6.5. *There is an $a < \min(t_1, t_2)$ and an $\eta < \bar{\eta}$ so that $|x(a)| > \eta$ and $|x(a-1)| > \eta$.*

We will prove that $V(x_a) \le 2k-1$. Then, since $V(x_t)$ decreases along solutions, we have $V(x_t) \le 2k-1$ for all t and so $x \in M_k$.

Proposition 6.6. $V(x_a) \leq 2k-1$

The proof requires a string of inequalities. First, we compare the number of sign changes in the vector associated with the ODE $(1.1)_n$ with the number of sign changes in the discretization of x^n. In particular, define

$$X^n := \langle x^n(a), \int_{-\infty}^0 x^n(a+s)r_1^n(s)ds, ..., \int_{-\infty}^0 x^n(a+s)r_n^n(s)ds\rangle$$

and

$$D^n := \langle x^n(a), x^n(a), x^n(a-\frac{1}{n}), x^n(a-\frac{2}{n}), ..., x^n(a-1+\frac{1}{n})\rangle$$

We expect the vectors X^n and D^n to be close since r_k^n has it's maximum at $-(k-1)/n$ and each r_k^n looks more and more like a δ- function as n gets bigger. We have

Proposition 6.6a. *For n large enough $\tilde{V}_n(X^n) \geq \tilde{V}_n(D^n)$.*

Next we compare the sign changes in the discretization vector D^n with the sign changes of the function x^n in the interval $(a-1, a]$. In particular, we want to show that if the discretization is fine enough then we have captured all of the sign changes of x^n. To this end for any function $y \in \cup_{n \leq \infty} \mathcal{A}_n$, define

$$N(y) := \text{ the number of sign changes of y in } (-1,0] + \frac{\text{sgn }(y(0)y(-1))+1}{2}$$

Then

Proposition 6.6b. *If n is large enough then $\tilde{V}(D^n) = N(T_n(a)x^n)$.*

Next we compare the functions x^n with the limit x.

Proposition 6.6c. *If n is large enough then $N(T_n(a)x^n) = N(T(a)x) = N(x_a)$.*

And finally

Proposition 6.6d. $N(x_a) = V(x_a)$

For Proposition 6.6 to hold, n must be large enough so that the following conditions hold (where η was chosen in Lemma 6.5).
C1. $\sup_{t \in [a-1,a]} |x^n(t) - x(t)| < \eta/4$
C2. $|x^n(t)| + |\dot{x}^n(t)| > C/2$ for all $t \in [a-1, a]$
C3. $|\int_{-\infty}^0 x(a+s)r_k^n(s)ds - x(a-(k-1)/n)| < \eta/4$ for all $0 \leq k \leq n$
C4. $1/n < d/2$

C5. $n > 2M/\eta$, where M is the bound on the magnitude of the derivative for functions in $\cup\mathcal{A}_n$.

That we can choose n large enough to satisfy 1 is a result of Lemma 4.1 and that we can satisfy 2 is a result of Lemmas 4.1 and 4.5 and Corollary 3.1b part 1. Condition 3 is possible by Remark 4.4. Clearly, we can choose n large enough to satisfy 4 and 5. These five constraints are technical and will show up in the proofs of the inequalities.

We begin with the proof of Proposition 6.6a.

Proof of 6.6a. We will show that we can add a perturbation to D^n of size less than $\eta/2$ without decreasing the number of zeroes. For this to be true, it must be the case that between any two consecutive zeroes of D^n, there must be a vector element with magnitude greater than $\eta/2$. Indeed, if for some k, we have

$$|x^n(a - \frac{k-1}{n})| > \eta/2 \tag{6.1}$$

then, provided C1 and C3 are satisfied above,

$$\begin{aligned}
&|x^n(a - \frac{k-1}{n}) - \int_{-\infty}^{0} x^n(a+s)r_k^n(s)ds| \\
&\leq |x^n(a - \frac{k-1}{n}) - x(a - \frac{k-1}{n})| + |x(a - \frac{k-1}{n}) - \int_{-\infty}^{0} x^n(a+s)r_k^n(s)ds| \\
&\leq \frac{\eta}{4} + \frac{\eta}{4} = \frac{\eta}{2}
\end{aligned}$$

and so

$$\int_{-\infty}^{0} x^n(a+s)r_k^n(s)ds > 0.$$

Hence the vector X^n cannot have fewer sign changes than D^n provided (6.1) holds.

Since x^n is continuous and $x^n \not\equiv 0$, x_n must reach an extreme value between consecutive zeroes. Since x^n satisfies $(1.3)_n$ and since f is continuous, x^n is differentiable at the extreme values and its derivative is 0, so by C2 above, the extreme values of x^n have magnitude at least $C/2$. Since $\eta < C/2$, we know that on some interval (which also contains the extreme point), $|x^n(t)| > \eta/2$. We must determine the minimum length of this interval. This minimum length is given by the "time" it takes for x^n to go from $\eta/2$ up to $C/2$ and back down again with maximum slope M. This time is $2(C/2 - \eta/2)/M = (C-\eta)/2$. Hence we require that the discretization interval be less than $(C-\eta)/2$; in that way we can be sure that whenever x_n achieves an extreme value, the discretization picks up some nearby point with value greater than $\eta/2$. This holds since if n is chosen as above, then we have

$$\frac{1}{n} < \frac{\eta}{2M} < \frac{C}{4M} < \frac{C}{2M} < \frac{C-\eta}{M}$$

This proves (6.1).

To finish the proof of Proposition 6.6a, we must also show that the crossterms keep the same value. The first term of each vector is the same so we need only check that

$$|\int_{-\infty}^{0} x^n(s+a)r_n^n(s)ds - x_n(a-1)| < \eta/2$$

but

$$|\int_{-\infty}^{0} x^n(s+a)r_n^n(s)ds - x^n(a-1)| \leq |\int_{-\infty}^{0} x^n(s+a)r_n^n(s)ds - x(a-1)|$$
$$+ |x(a-1) - x_n(a-1)| \leq \eta/4 + \eta/4 = \eta/2$$

So $\int_{-\infty}^{0} x^n(s+a)r_n^n(s)ds$ has the same sign as $x^n(a-1)$. Since $|x(a-1)| > \eta$, then $|x^n(a-1)| > \eta/2$. If x^n has maximum slope M, then at worst

$$|x^n(a-1+1/n) - x^n(a-1)| < M/n < \eta/2$$

Hence $x(a-1+1/n)$ has the same sign as $x(a-1)$ and so the crossterms are the same. ■

Proof of Proposition 6.6b. In order to catch all the sign changes of x^n in D^n, we must choose n so that $1/n$ is smaller than the minimum distance between zeroes of x^n and so that there are no zeroes in the interval $(a-1, a-1+1/n]$. We must also show that

$$\frac{\operatorname{sgn}((x^n(a)x^n(a-1))+1}{2} = \frac{\operatorname{sgn}(x^n(a)x^n(a-1+1/n))+1}{2} \tag{6.2}$$

Using conditions C1 and C2 above, we can show that, given our choice of η, the minimum distance between consecutive zeroes of x^n is $d/2$ (we treat the interval $(a-1, a-1+1/n]$ separately in the next paragraph). Let $z_1 < z_2$ be consecutive zeroes of x in the interval $(a-1+1/n, a]$. For $i = 1, 2$, let a_i be the first point to the left of z_i for which $|x(t)| = \eta/2$ and let b_i be the first point to the right of z_i for which $|x(t)| = \eta/2$. We know that a_2 and b_1 exist since $|x(t)|$ must exceed $C > \eta$ somewhere in $[z_1, z_2]$, and we know that a_1 and b_2 exist since $|x(a-1)| > \eta$ and $|x(a)| > \eta$. In each interval $[a_i, b_i]$, x is monotone since $\dot{x}(t)$ can only be zero if $|x(t)| > C$. By C1, we must have that in each interval $[a_i, b_i]$, $|x^n(t)| < \eta$. Hence x^n is also monotone in $[a_i, b_i]$ since, by C2, $\dot{x}^n(t)$ can only be zero if $|x^n(t)| > C/2 > \eta$. Also by C1, $x^n(a_i)$ and $x_n(b_i)$ have the same signs as $x(a_i)$ and $x(b_i)$. Therefore, x^n also has exactly one zero in each interval $[a_i, b_i]$ and x^n has no zeroes in $[a_2, b_1]$ so these zeroes are consecutive. The minimum distance between these zeroes is $a_2 - b_1$.

Now we use what we know about x to get an estimate on $a_2 - b_1$. When $t \in [z_1, b_1]$ or $t \in [a_2, z_2]$, we have $|x(t)| < \eta/2$ and so in this interval the minimum slope of x is $|\dot{x}(t)| > C - |x(t)| > C - \eta/2$. A short computation shows that $b_1 - z_1 > \eta/(2C - \eta)$ and

$z_2 - a_2 > \eta/(2C-\eta)$. So

$$\begin{aligned} a_2 - b_1 &= (z_2 - z_1) - (b_1 - z_1) - (z_2 - a_2) \\ &> d - \frac{2\eta}{2C-\eta} \\ &> d - \frac{2(2dC/(d+4))}{2C - (2dC/(d+4))} \\ &= d - \frac{d}{2} \\ &= \frac{d}{2} \end{aligned}$$

where we have taken advantage of our choice of η.

Since, according to Lemma 6.5 and C1. above, we have $|x^n(a-1)| > \eta/2$, the closest zeroes of x^n to $a-1$ must be at least $\eta/2M$ units away. Since we have chosen $\bar{n} > 2M/\eta$ there cannot be a zero of x^n in the interval $[a-1, a-1+1/n]$. Since there are no zeroes in this interval, the terms $x^n(a-1)$ and $x^n(a-1+1/n)$ have the same sign and (5.2) holds. ∎

Proof of Proposition 6.6c. Since $|x(a)|, |x(a-1)| > \eta$ and $|x(t) - x^n(t)| < \eta/4$, we know that $x^n(a)$ and $x(a)$ have the same sign and $x^n(a-1)$ and $x(a-1)$ have the same sign. According to the proof of proposition 6.6b, both x^n and x are monotonic when they take values in the strip $[-\eta/2, \eta/2]$ and extreme values must lie outside this strip. Hence, again by condition C1, the number of sign changes in $(a-1, a]$ must be the same for both. ∎

Proof of Proposition 6.6d. We will consider the cases $V(x_a) = 1$ and $V(x_a) \geq 3$ separately (remember that V is always odd).

Suppose $V(x_a) = 1$. Remember that there are two parts to $N(x_a)$, the number of zeroes of x_a and the crossterm. We begin by proving that in the interval $(a-1, a]$, x may have either one zero or no zeroes (remember that if σ does not exist then we set $V = 1$). Suppose there are $j > 1$ zeroes. Call the zero which is closest to a on the left $\bar{\sigma}$. Then $V(x_{\bar{\sigma}}) \geq j$, but this contradicts the fact that $V(x_t) = 1$ for all $t \in (-\infty, t_1]$, so $j \leq 1$. Now we compute the crossterm. If $V(x_a) = 1$ and $j = 1$, then $\mathrm{sgn}(x(a)x(a-1)) = -1$, so $N(T(a)x) = 1 = V(x_a)$. If $j = 0$, then $\mathrm{sgn}(x(a)x(a-1)) = +1$ and again $N(T(a)x) = 1 = V(x_a)$.

Now suppose $V(x_a) = m \geq 3$ and let j be the number of zeroes of x in $(a-1, a]$. We begin by proving that either $j = m$ or $j = m-1$. Suppose instead $j \leq m-2$. Then $m = V(x_a) \leq j+1 \leq m-1$, so this cannot be. Now suppose $j \geq m+1$. Again we call the zero which is closest to a on the left $\bar{\sigma}$. Then $V(x_{\bar{\sigma}}) \geq j \geq m+1$, but this contradicts the fact that $V(x_t) = m$ for all $t \in (-\infty, t_1]$. Again, we must check the contribution made by the crossterms. If $j = m-1$, then, since $m-1$ is even, $\mathrm{sgn}(x(a)x(a-1)) = +1$ and so

$N(T(a)x) = m = V(x_a)$. If there are m zeroes, then since m is odd, $\mathrm{sgn}(x(a)x(a-1)) = -1$ and $N(T(a)x) = m = V(x_a)$. ■

Since $x^n \in S_k^n$ for all n we have $\tilde{V}_n(X^n) = 2k-1$ for all n. If we apply the inequalites in Propositions 6.6a through 6.6d, we get

$$V(x_a) = N(x_a) = N(T_n(a)x^n) = \tilde{V}_n(D^n) \leq \tilde{V}_n(X^n) = 2k-1$$

and so we have proved Proposition 6.6.

Lemma 6.7. *Assumption 6.4 holds.*

Proof

Suppose $x^n \to x$ in X and $|x(t)| \to 0$ as $t \to -\infty$. Define $L < 0$ so that for all $t \in (-\infty, L]$, $|x(t)| < \zeta/4$. Define

$$q_n = \sup\{t < L : |x^n(t)| = \zeta/2\}$$

We want to show that there is a subsequence $q_{n_i} \to \infty$. Suppose instead that there is a $Q > -\infty$ such that $q_n \geq Q$ for all n. Let $\tilde{q} < Q$. Pick k so that $L - 2k < \tilde{q} < L - 2(k-1)$. Consider $T_n(L)x^n, T_n(L-2)x^n, ..., T_n(L-2k)x^n$. For each j there exists N_j so that for all $n > N_j$

$$\|T_n(L-2j)x^n - T(L-2j)x\|_X < \zeta/4$$

and so

$$|[T_n(L-2j)x^n](\theta) - [T(L-2j)x](\theta)| < \zeta/4$$

for all $\theta \in [-2, 0]$ and $j = 0, \ldots, k$. Hence

$$|[T_n(L-2j)x^n](\theta)| < \zeta/2$$

for all $\theta \in [-2, 0]$ and $j = 0, \ldots, k$. Choose $N = \max_j N_j$. Then for all $n > N$, we have

$$|x^n(t)| < \zeta/2$$

for $t \in [L-2k, L]$. Hence for $n > N$, $q_n < \tilde{q}$ and we have reached a contradiction. So such a subsequence exists. For conveniemce, we also call the subsequence $\{q_n\}$.

Now define

$$y^n(t) = x^n(t + q_n)$$

for all t. Since S_k^n is invariant, $y^n \in S_k^n$ for all n. There is a subsequence, which we again call y^n, so that $y^n \to y$ in X. By Lemma 6.1, $y \in \mathcal{A}_\infty$. We want to show that $|y(t)| < \zeta$ for all $t > 0$. Suppose there is a $\tilde{t} > 0$ so that $|y(\tilde{t})| = \zeta$. Consider $T(\tilde{t})y$. $T_n(\tilde{t})y_n \to T(\tilde{t})y$ in X so there exists an $\tilde{N}$ so that for all $n > \tilde{N}$,

$$\|T_n(\tilde{t})y_n - T(\tilde{t})y\|_X < \zeta/2$$

and so

$$|[T_n(\tilde{t})y_n](0) - [T(\tilde{t})y](0)| < \zeta/2$$

Pick $N \geq \tilde{N}$ so that for all $n > N$, $L - q_n > \tilde{t}$. Then for all $n > N$, $|[T_n(\tilde{t})y_n](0)| < \zeta/2$ and so

$$|[T(\tilde{t})y](0)| < \zeta$$

and we have arrived at a contradiction. Hence $|y(t)| < \zeta$ for all $t > 0$.

Therefore, by Corollary 3.3, $y \in W^s(0)$. If also $y(t) \to 0$ as $t \to -\infty$, then y would be a homoclinic orbit, but according to [M-P] this is impossible. Since $y(t) \not\to 0$ as $t \to -\infty$, Assumption 6.4 holds for and we use Theorem 1.1 to conclude that

$$N(y) \leq 2k - 1$$

However according to Corollary 7.2 of [M-P], if $y \in W^s(0) \setminus \{0\}$, then $N(y) > 2N^* + 1$ and so we have a contradiction. This proves the Lemma. ■

We close this section with a proof of Remark 1.0.

Proof of Remark 1.0. Let $a \geq u$ and let $B_a^n \subset I\!R^{n+1}$ be the box centered at the origin with side $2a$. Observe that the vector field associated with $(1.1)_n$ points inwards on all sides of the box except maybe on the edges $x_1 = a$ and $x_1 = -a$. The assumption however guarantees that the vector field also points inwards on these two hyperplanes. Hence $\tilde{\mathcal{A}}_n \subset B_u^n$ for all n.

Notice that this does not mean that the attractors $\tilde{\mathcal{A}}_n$ are uniformly bounded as $n \to \infty$ even though u does not depend on n. Indeed, the diagonal of B_u^n grows as $\sqrt{n+1}$ and so there is no fixed bounded set which contains all the attractors $\tilde{\mathcal{A}}_n$.

However, we want to show boundedness in X. Let $x \in \mathcal{A}_n \subset X$ for some n. Then $[L_n x](t) \in \tilde{\mathcal{A}}_n$ for all t and $[L_n x](t) \in B_u^n$. So

$$\begin{aligned} \|x\|_X &= \sup_{-2 \leq s \leq 0} |x(s)| + \int_{-\infty}^{0} x(s) Q_1(s) ds \\ &\leq u + u \int_{-\infty}^{0} Q_1(s) ds \\ &= 2u \end{aligned}$$

■

7. Proof of Theorem 1.2b

In this section, we consider the special case

$$\dot{x}(t) = f(x(t-1)). \tag{7.1}$$

In this case, the solutions satisfy assumption **A5** (this is shown in Lemma 7.14 at the end of this section).

The proof of part a of the Theorem is given in section 5. To prove part b, we begin just as in the proof of Theorem 1.1b. We consider a sequence $x^n \in S_k^n$. There is a convergent subsequence $x^{n_j} \to x$. We will show that $x \in S_k^\infty$ and then we invoke Lemmas 6.1 through 6.3 to complete the proof.

As in the last section, we will consider these functions on a fixed interval $[a-1, a]$. We will choose $a > t_5 + 2L + 1$ where L is as in the Lemma

Lemma 7.1. *There exists an $L > 0$ so that every interval of length L contains a zero of x, provided $x \in S_k^\infty$ for some k.*

and t_5 is chosen as follows. Since the stable manifold of the origin is not included in $\mathcal{A}_\infty$, $x(t)$ does not converge to zero as $t \to \infty$. Hence by Theorem 3.1a, there is a t_1 such that all of the zeroes of x which lie in (t_1, ∞) are simple. This implies that also the zeroes of $\dot{x}(t)$ are simple in the interval $(t_1 + 1, \infty)$ since if $\dot{x}(s) = \ddot{x}(s) = ... = x^{(k)}(s) = 0$, then $x(s-1) = \dot{x}(s) = ... = x^{(k-1)}(s) = 0$. According to the assumption **A5**, there is a $t_2 > 0$ such that for $t \in (t_2, \infty)$, there is exactly one zero of $\dot{x}(t)$ between two consecutive zeros of x. Let t_3 be the first zero of x which is larger than t_2. Since x has zeroes in (t_3, ∞), we must also have, in (t_3, ∞), that between any two zeroes of $\dot{x}$ there be a zero of x. Otherwise, when x finally reaches zero again, there will have been many values of t for which $\dot{x}(t) = 0$. Finally, there is a t_4 so that $V(x_t)$ is constant for $t > t_4$. Then $t_5 = \max(t_1 + 1, t_3, t_4)$.

In the proof of Theorem 1.2b, we will take advantage of the fact that if $\dot{x}(t) = 0$ then $x(t-1) = 0$. We introduce some notation for x^n and its derivatives. For $i > 0$ let

$$x_i^n(t) := \int_{-\infty}^{0} x^n(t+s) r_i^n(s) ds$$

and let $x_0^n(t) := x^n(t)$ so that $x_i^n(a)$ is the $(i+1)$st element of the vector X^n. Let $y_i^n(t) := \dot{x}_i^n(t)$ and let Y^n be the vector whose $(i+1)$st element is $y_i^n(a)$. We will also use y^n for y_0^n. Finally, let $y(t) := \dot{x}(t)$.

The proof of the Theorem will require several results about the placement of zeroes of y_k^n. We begin with the Proof of Lemma 7.1.

Proof of Lemma 7.1. The statement was proved in [M-P] by J. Mallet-Paret for the case $k = 1$. The case $k > 1$ is simple. Suppose, without loss of generality, that $x(0) = 0$. We claim that the next zero of x is in $(0, 1]$. Suppose instead that $z > 1$ is the next zero. Then $V(x_z) = k$. If z is simple, then there must be $k - 1$ zeroes in $(z-1, z]$ but this contradicts the assumption that there are no zeroes in $(0, z)$. If z is not simple, then $z - 1$

must be a zero of one less order than z (Lemma 5.2,[M-P]), but this again contradicts that there are no zeroes in $(0, z)$. ■

Next we obtain a bound on $\ddot{y}$.

Lemma 7.2. *There exists an $H > 0$ so that for all $x \in \mathcal{A}_\infty$ and for all t, we have $|\ddot{y}| \leq H$.*

Proof. We compute $\ddot{y}$.

$$\ddot{y}(t) = f''(x(t-1))\dot{x}(t-1)^2 + f'(x(t-1))^2\dot{x}(t-1)$$

¿From Proposition 3.4 and Lemma 4.2, we know that there are K and M so that for all $x \in \mathcal{A}_\infty$ and for all t, $|x(t)| \leq K$ and $|\dot{x}(t)| \leq M$. Hence there must also be an H so that for all $x \in \mathcal{A}_\infty$ and for all t, $|\ddot{y}(t)| \leq H$. ■

Next we show that the functions y_i^n must have zeroes near zeroes of $y_{-(i-1)/n}$ where by this notation we mean the function defined by $y_{-(i-1)/n}(t) = y(t-(i-1)/n)$. To do this we begin by proving that, if z is a zero of y, then both y and y^n are monotone in a fixed interval containing z. The same holds for y_i^n and $y_{-(i-1)/n}$. This is proved in Lemma 7.3. In Lemmas 7.4 and 7.5 we prove that the zeroes are indeed close, since y_i^n is a small enough perturbation of $y_{-(i-1)/n}$. In the Lemmas, we use the fact that y^n, $\dot{y}^n$ and $\ddot{y}^n$ satisfy the hypotheses of Lemma 4.3. This is shown by applying Lemma 4.2 and Proposition 4.6. We will use $J := [t_5 + 2L, t_5 + 3L + 1]$.

Lemma 7.3. *If $z \in J$ is a zero of y, H is as above, $B = f'(0) < 0$ and C is as in Theorem 3.1, then y is monotone in the interval $[z + \frac{BC}{H}, z - \frac{BC}{H}]$. Furthermore if n is large enough that*

$$\sup_{t \in J} |\dot{y}_0^n(t) - \dot{y}(t)| < \frac{|BC|}{2} \qquad (i)_0$$

and

$$\sup_{t \in J} |\ddot{y}_0^n(t) - \ddot{y}(t)| < H \qquad (ii)_0$$

then y^n is monotone in $[z + \frac{BC}{4H}, z - \frac{BC}{4H}]$. If n is large enough so that for $1 \leq i \leq n$

$$\sup_{t \in J} |\dot{y}_i^n(t) - \dot{y}(t - \frac{i-1}{n})| < \frac{|BC|}{2} \qquad (i)_n$$

and

$$\sup_{t \in J} |\ddot{y}_i^n(t) - \ddot{y}(t - \frac{i-1}{n})| < H \qquad (ii)_n$$

then y_i^n is monotone in $[z - \frac{i-1}{n} + \frac{BC}{4H}, z - \frac{i-1}{n} - \frac{BC}{4H}]$.

Proof. Since $\dot{x}(z) = 0$, also $x(z-1) = 0$. Then, since from Theorem 3.1 $|\dot{x}(z-1)| = |x(z-1)| + |\dot{x}(z-1)| > C$, we have

$$\begin{aligned} \dot{y}(z) = \ddot{x}(z) &= f'(x(z-1))\dot{x}(z-1) \\ &\geq BC. \end{aligned} \tag{7.2}$$

Since $|\ddot{y}(t)| \leq H$, we must have $|\dot{y}(t)| > 0$ for $t \in [z + BC/H, z - BC/H]$ and so in this interval y is monotone.

If n is large, then by $(i)_0$ and (7.2), $|\dot{y}_0^n(z)| > |BC|/2$ and by $(ii)_0$, $|\ddot{y}_0^n(t)| \leq 2H$ for $t \in J$ and so $|\dot{y}_0^n| > 0$ in $[z + BC/4H, z - BC/4H]$ and so y_0^n is monotone. The argument for y_j^n is the same. ■

Suppose that y has ℓ zeroes in J, namely $z_1, ..., z_\ell$. Define

$$\bar{\eta} = \min_{1 \leq j \leq \ell} \left(|y(z_j + \frac{BC}{8H})|, |y(z_j - \frac{BC}{8H})| \right)$$

Since the distance between zeroes of y and $\dot{y}$ is at least $|BC|/H$ by Lemma 7.3, the distance between zeroes of y is also at least $|BC|/H$, so $\bar{\eta} > 0$.

Lemma 7.4. *Let $\epsilon < \bar{\eta}$. If the assumptions of Lemma 7.3 are satisfied and if N is large enough so that for all $n > N$,*

$$\sup_{t \in J} |y_i^n(t) - y_{-(i-1)/n}(t)| < \epsilon/2$$

for $0 \leq i \leq n$, then, for $n > N$, y^n has exactly one zero in $[z + \frac{BC}{4H}, z - \frac{BC}{4H}]$ and this zero is in $[z + \frac{BC}{8H}, z - \frac{BC}{8H}]$. y_i^n has exactly one zero in $[z - \frac{i-1}{n} + \frac{BC}{4H}, z - \frac{i-1}{n} - \frac{BC}{4H}]$ and this zero is in $[z - \frac{i-1}{n} + \frac{BC}{8H}, z - \frac{i-1}{n} - \frac{BC}{8H}]$.

Proof. The proof follows from the observations that y_0^n is monotone in $[z + BC/4H, z - BC/4H]$ and $|y(z + BC/4H)| > |y(z + BC/8H)| > \epsilon$ and $|y(z - BC/4H)| > |y(z - BC/4H)| > \epsilon$. The proof for y_i^n is the same. ■

If we call the zero of y_i^n in Lemma 7.4 z_i^n, then finally we have

Lemma 7.5. *If the assumptions of Lemma 7.3 are satisfied, then for all $\bar{\eta} > \epsilon > 0$ there is an N so that for all $n > N$*

$$|z_i^n - z - \frac{i-1}{n}| < \epsilon$$

.

Proof. Choose N large enough so that according to Lemma 7.4. for all $n > N$,

$$\sup_{t \in J} |y_i^n(t) - y_{-(i-1)/n}(t)| < \max(\frac{\epsilon}{4}|BC|, \frac{\bar{\eta}}{2}).$$

Since $|\dot{y}_{-(i-1)/n}(z-(i-1)/n)| \geq |BC|$ by equation (7.2) and since $(i)_n$ holds, we know that $|\dot{y}_i^n(z-(i-1)/n)| > BC/2$. By Lemma 7.2 and $(ii)_n$, we know $|\ddot{y}_i^n(t)| \leq 2H$ for all $t \in J$. Hence $|\dot{y}_i^n(t)| > BC/4$ for $t \in [z - \frac{i-1}{n} + \frac{BC}{8H}, z - \frac{i-1}{n} - \frac{BC}{8H}]$. This last interval is the i nterval containing z_i^n according to Lemma 7.4. Then

$$\frac{|BC|}{4} < \left| \frac{y_i^n(z-\frac{i-1}{n}) - y_i^n(z_i^n)}{z_i^n - z + \frac{i-1}{n}} \right| = \left| \frac{y_i^n(z-\frac{i-1}{n})}{z_i^n - z + \frac{i-1}{n}} \right|$$

So

$$\begin{aligned} |z_i^n - z + \frac{i-1}{n}| &< \frac{4}{|BC|}|y_i^n(z-\frac{i-1}{n})| \\ &= \frac{4}{|BC|}|y_i^n(z-\frac{i-1}{n}) - y_{-(i-1)/n}(z-\frac{i-1}{n})| \\ &< \epsilon \end{aligned}$$

■

The convergence of zeroes of y_i^n will be key in the proof of the Theorem. Before the proof though, we must choose a. Choose any $\eta < \bar{\eta}$ and $a \in [t_5 + 2L + 1, t_5 + 3L + 1]$ so that

a1. $|x(a)| > \eta, \quad |x(a-1)| > \eta$
a2. $|\dot{x}(a)| > \eta, \quad |\dot{x}(a-1)| > \eta$
a3. The number of zeroes of $\dot{x}$ in $[a-1, a]$ is strictly less than the number of zeroes of x in $[a-1, a]$.

The simple proof in Lemma 6.4 shows that it is possible to choose a to satisfy a1, but we must show that all three conditions can be satisfied simultaneously. In fact, this is almost as simple.

Lemma 7.6. *There exists an a and η so that a1 through a3 are satisfied.*

Proof of Lemma 7.6 Choose $z > t_5 + 2L + 1$ so that $x(z) = 0$. Since the zeroes are simple, $\dot{x}(z) \neq 0$. Let q be the zero of $\dot{x}$ just to the right of z. From the differential equation, we know that $x(q-1) = 0$. Choose $a \in (z, q)$ so that $a-1$ is close enough to $q-1$ that $\dot{x}$ has no zeroes in $[a-1, q-1]$. This is possible since $\dot{x}(q-1) \neq 0$. Then $\dot{x}$ has no zeroes in $[a-1, q-1] \cup [z, a]$, and in $[q-1, z]$, $\dot{x}$ must have one fewer zero than x. Hence in the entire interval $[a-1, a]$, $\dot{x}$ has one fewer zero than x. Since none of $x(a)$, $x(a-1)$, $\dot{x}(a)$ and $\dot{x}(a-1)$ are zero, we pick $\eta < \min(|x(a)|, |x(a-1)|, |\dot{x}(a)|, |\dot{x}(a-1)|)$. ■

We will prove that, for large enough n, $\tilde{V}_n(X^n) \leq N(x_a) + 1$, where X^n is as defined in section 6. By proposition 6.6d, this is the same as $\tilde{V}_n(X^n) \leq V(x_a) + 1$. With this result and Theorem 1.1b, we have then

$$2k - 2 = \tilde{V}_n(X^n) - 1 \leq V(x_a) \leq \tilde{V}_n(X^n) = 2k - 1$$

and since V takes only odd integer values, we have

$$V(x_a) = 2k - 1$$

and so $x \in S_k^\infty$.

Proposition 7.7. *For large n, $\tilde{V}_n(X_n) \leq N(x_a) + 1$.*

We define the following functionals which are related to the Lyapunov functions and the functional N defined in section 6, but do not include the "crossterm". Define $Z : C([a-1,a], I\!R) \to I\!N$ as

$$Z(\phi) := \text{ the number of zeroes of } \phi \text{ in } [a-1, a]$$

and define $W : I\!R^{n+1} \to I\!N$ as

$$W(\mathbf{v}) := \text{ the number of sign changes in } \mathbf{v}.$$

Lastly, we define C^n to be the crossterm associated with the vector X^n.

$$C^n = \frac{\text{sign } (x_n^n(a)x_1^n(a)) + 1}{2}.$$

Then we have the following inequalities.

Proposition 7.7a. $\tilde{V}_n(X^n) - 1 - C^n \leq W(Y^n)$.

Proposition 7.7b. *For large enough n, $W(Y^n) \leq Z(y^n)$.*

Proposition 7.7c. *For large enough n, $Z(y^n) = Z(y)$.*

Proposition 7.7d. $Z(y) = Z(\dot{x}) < N(x_a) + 1 - C^n$.

Proposition 7.7d is simply the result of the choice of a which says $Z(\dot{x}) < N(x_a)$ (remember that C^n is either 0 or 1). Since Z takes integer values, 7.7d becomes $Z(\dot{x}) \leq N(x_a) - C^n$. So from the four inequalites we get $\tilde{V}_n(X^n) - 1 - C^n \leq N(x_a) - C^n$ or $\tilde{V}_n(X^n) \leq N(x_a) + 1$.

We do the remaining proofs in the order 7.7a, 7.7c and 7.7b.

Proof of Proposition 7.7a. Suppose that we have two consecutive sign changes in the vector X^n. Then, without loss of generality, we can assume that there are integers $i < j$ such that $x_{i-1}^n(a) < 0$, $x_i^n(a) > 0$, $x_{j-1}^n(a) > 0$ and $x_j^n(a) < 0$. Then

$$y_i^n(a) = \dot{x}_i^n(a) = n(x_{i-1}^n(a) - x_i^n(a)) < 0$$
$$y_j^n(a) = \dot{x}_j^n(a) = n(x_{j-1}^n(a) - x_j^n(a)) > 0$$

and so between any two sign changes in the vector X^n there is a sign change in the vector Y^n and so the number of sign changes in X^n is at most one more than the number of sign changes in Y^n. Since

$$\tilde{V}_n(X^n) = \# \text{ of sign changes in } X^n + C^n$$

we are done. ■

Proof of Proposition 7.7c. For the most part, this is just the result of Lemma 7.4. If z is a zero of y, then y^n has exactly one zero in $[z + \frac{BC}{4H}, z - \frac{BC}{4H}]$. Note that there are no other zeroes of y in this interval; that is, in $[z + \frac{BC}{4H}, z - \frac{BC}{4H}]$, both y and y_k^n have exactly one zero. The only thing that we need to show then in order to prove the Theorem is that if $z + \frac{BC}{2H} < a - 1$ then the zero of y_k^n is in $[a-1, z]$ and that if $z - \frac{BC}{2H} > a$, then the zero of y_k^n is in $[z, a]$. But the proof follows from the proof of Lemma 7.4 since both $|y(a-1)| > \eta$ and $|y(a)| > \eta$. ■

Next we prove Proposition 7.7b. In order to prove the Proposition, we must construct a special sequence of zeroes associated with the functions x^n.

Let σ be the largest zero of y which is less that $a - 1$. Every interval of length L contains a zero of x, and the same must be true for $\dot{x}$, so $\sigma > a - 1 - L$ and so it is a simple zero. According to Lemma 7.4, y_0^n has exactly one sign change in $[\sigma + BC/8H, \sigma - BC/8H]$. We will call this zero $s_0^0(n)$. Clearly these zeroes are also simple.

We define zeroes $s_0^i(n) \geq s_0^0(n)$ of y_i^n as follows. Let $s_0^i(n)$ be the first zero larger than $s_0^{i-1}(n)$ such that $\dot{y}_i^n(s_0^i(n))\dot{y}_{i-1}^n(s_0^{i-1}(n)) > 0$. Then we define further zeroes of y_i^n. Let $s_j^i(n)$ be the jth zero of y_i^n larger than $s_0^i(n)$.

Since y is not zero in the interval $[\sigma - BC/8H, a]$, we can choose N_1 so that for $n > N_1$, y_0^n is also never zero in this interval. Hence, for $n > N_1$, $s_0^1(n) > a$.

Lemma 7.8. *There is N_2 so that for all $n > N_2$, all the zeroes $s_j^i(n)$ which are in J are simple for all i and j.*

Proof. This is a corollary of Lemmas 7.3 and 7.4. The zeroes $s_j^i(n)$ converge to a zero of $y_{-(i-1)/n}$ as $n \to \infty$; call it z. Let $\epsilon < \min(|BC|/2, \bar{\eta})$. According to Lemma 7.4, there is an N so that if $n > N$, then y_i^n has exactly one zero, $s_j^i(n)$, in $[z - \frac{j-1}{n} + \frac{BC}{4H}, z - \frac{j-1}{n} + \frac{BC}{4H}]$. According to Lemma 7.3, if n is large enough, then in this interval $|\dot{y}_j^n(t)| > 0$. ■

Now we consider the sequences $s_j^i(n)$ of zeroes of y_i^n.

Let $n > N_3 := \max(N_1, N_2)$. Let b be the first zero of y_0^n which is greater than a and let u be the number of zeroes of y_0^n in $(s_0^0(n), b]$. Then let $\alpha(i)$ be the number of sign changes in the vector of length $i+1$, $(y_0^n(a), y_1^n(a), ...y_i^n(a))$ and let $\beta(i)$ be the number of zeroes of y_i^n in $(s_0^i(n), a]$ provided $s_0^i(n) < a$.

The proof of Proposition 7.7b will then be a corollary of the following.

Lemma 7.9. *There is an $N > 0$ so that for all $n > N$ and every $0 \le i \le n$, $s_0^i(n) < a$.*

Proposition 7.10. *For $n > N$ and $0 \le i \le n$, $\alpha(i) + \beta(i) \le u - 1$*

¿From Proposition 7.10, we can conclude that $\alpha(n) \le u-1$. Since there are no zeroes of y_0^n in $(s_0^0(n), a-1]$ and in $[a, b)$, $u-1$ is exactly the number of zeroes of y_0^n in $[a-1, a]$. So $\alpha(n) \le u-1$ is really the same as the statement in Proposition 7.7b.

Now we will prove Lemma 7.9 and Proposition 7.10.

Proof of Lemma 7.9. Since $s_0^i(n) \le s_0^n(n)$, it is enough to show that $s_0^n(n) < a$. Since σ is a zero of the function y, for $i \ge 1$, there is a corresponding zero $\sigma_i^n := \sigma + (i-1)/n$ of the function $y_{-(i-1)/n}$. From Lemma 7.5, we know that y_i^n has a zero close to this. We will call this zero a_i^n. We will show that, for large n, $a_i^n = s_0^i(n)$. In that case, $s_0^n(n)$ is close to $\sigma + 1 < a$ and the Lemma will be proved.

For each n, we do the proof by induction on i. By definition, $a_0^n = s_0^0(n)$. Now assume that $a_{i-1}^n = s_0^{i-1}(n)$. Pick

$$\delta < \min(\bar{\eta}, |BC|/16H, |\sigma - (a-1)|)$$

and

$$N_5 > \max(8H/|BC|, N_3, N_4)$$

where N_4 is the N in Lemma 7.5 with $\epsilon = \delta$ and N_3 is as above. Then for $n > N_5$,

$$2\delta + 1/n < |BC|/4H.$$

Let $n > N_5$.

According to Lemma 7.3, the distance between consecutive zeroes of y_i^n is at least $BC/4H$, for all i. Hence there can be only one zero of y_i^n in the interval

$$[\sigma_{i-1}^n - \delta, \sigma_i^n + \delta] = [\sigma_{i-1}^n - \delta, \sigma_{i-1}^n + 1/n + \delta]$$

since this interval has length $2\delta + 1/n$. In fact, we picked n large enough so that a_i^n is in

$$[\sigma_i^n - \delta, \sigma_i^n + \delta] \subset [\sigma_{i-1}^n - \delta, \sigma_i^n + \delta]$$

To be sure that this zero is $s_i^n(n)$ we need to check that $\dot{y}_i^n(a_i^n)\dot{y}_{i-1}^n(a_{i-1}^n) > 0$ and $a_i^n \ge a_{i-1}^n$. Let $\xi = \dot{y}(\sigma)$. By the convergence of $\dot{y}_i^n$ in Lemma 4.3 and by Lemma 7.5, we know that we can pick N_6 so that for all $n > N_6$,

$$\begin{aligned} |\dot{y}_i^n(a_i^n) - \dot{y}_{-(i-1)/n}(\sigma + \frac{i-1}{n})| &< \left|\frac{\xi}{4}\right| \\ |\dot{y}_{i-1}^n(a_{i-1}^n) - \dot{y}_{-(i-2)/n}(\sigma + \frac{i-2}{n})| &< \left|\frac{\xi}{4}\right| \end{aligned} \tag{7.3}$$

Hence, we choose $N := \max(N_5, N_6)$. Then

$$\begin{aligned}
\dot{y}_i^n(a_i^n)\dot{y}_{i-1}^n(a_{i-1}^n) &= \dot{y}_i^n(a_i^n)\dot{y}_{i-1}^n(a_{i-1}^n) - \dot{y}_{-(i-1)/n}(\sigma_i^n)\dot{y}_{-(i-2)/n}(\sigma_i^n) \\
&\quad + \dot{y}_{-(i-1)/n}(\sigma_i^n)\dot{y}_{-(i-2)/n}(\sigma_i^n) \\
&= \dot{y}_i^n(a_i^n)\dot{y}_{i-1}^n(a_{i-1}^n) - \dot{y}_i^n(a_i^n)\dot{y}_{-(i-2)/n}(\sigma_{i-1}^n) \\
&\quad + \dot{y}_i^n(a_i^n)\dot{y}_{-(i-2)/n}(\sigma_{i-1}^n) - \dot{y}_{-(i-1)/n}(\sigma_i^n)\dot{y}_{-(i-2)/n}(\sigma_i^n) + \xi^2 \\
&\geq -\frac{5\xi}{4}\frac{\xi}{4} - \frac{5\xi}{4}\frac{\xi}{4} + \xi^2 \\
&= \frac{6}{16}\xi^2 \\
&> 0
\end{aligned}$$

where we have used (7.3) and the fact that $|\dot{y}_{-(i-1)/n}(\sigma_i^n)| = \xi$ for all $1 \leq i \leq n$. The fact that $a_i^n \geq a_{i-1}^n$ is now a result of Lemma 7.11 which follows. And so $a_i^n = s_0^i(n)$. Hence the Lemma holds by induction. ∎

To prove Proposition 7.10, we require the following three lemmas. We will leave off the superscript n in these lemmas and their proofs.

Lemma 7.11. *Assume that sign y_{i-1} is constant (nonzero) in an interval $[s_1, s_2]$ and that $y_{i-1}(s_1)y_i(s_1) > 0$. Then $y_{i-1}(t)y_i(t) > 0$ for all $t \in [s_1, s_2]$. Simply put, y_i cannot change sign before y_{i-1}.*

Lemma 7.12. *Let $z_1 \in (t_2, \infty)$ and $z_2 \in (t_2, \infty)$ be consecutives zeroes of y_i (they are simple). Then y_{i-1} must have a zero in $[z_1, z_2]$.*

Lemma 7.13. *Let $b > s_0^n$. Then the number of zeroes of y_i in $(s_0^i, b]$ is nonincreasing in i.*

Proof of Proposition 7.10. We will do this proof by induction.
Let $i = 0$. Clearly $\alpha(0) = 0$ and $\beta(0) = u - 1$.

Now assume that for $i = j$, $\alpha(j) + \beta(j) \leq u - 1$. Notice that β is a nonincreasing function of i by Lemma 7.9, so $\beta(j+1) \leq \beta(j)$. Clearly either $\alpha(j+1) = \alpha(j)$ or $\alpha(j+1) = \alpha(j) + 1$. If $\alpha(j+1) = \alpha(j)$, then we are done. We will prove that when $\alpha(j+1) = \alpha(j) + 1$ then $\beta(j+1) < \beta(j)$.

Between s_0^{j+1} and $s_{\beta(j)}^{j+1}$, y_j must have at least $\beta(j)$ zeroes since y_j has a zero between any two zeroes of y_{j+1}. This means that $s_{\beta(j)}^j < s_{\beta(j)}^{j+1}$ since $s_0^j < s_0^{j+1}$. Suppose that $\beta(j) = \beta(j+1)$. Since $\alpha(j+1) = \alpha(j) + 1$, we know that the signs of $y_j(a)$ and $y_{j+1}(a)$ are different. So y_j and y_{j+1} must have opposite signs throughout the interval $(s_{\beta(j)}^{j+1}, a]$. and the same signs throughout the interval $(s_{\beta(j)}^j, s_{\beta(j)}^{j+1})$. But then since y_j has no sign changes in $[s_{\beta(j)}^j, a]$, y_{j+1} would have to change signs in $[s_{\beta(j)}^j, a]$ but then y_{j+1} would have to change sign before y_j and this is impossible by Lemma 7.7. ∎

Now we turn our attention to the three lemmas 7.11 through 7.13.

Proof of Lemma 7.11. Assume without loss of generality that $\dot{x}_{i-1}(t) = y_{i-1}(t) < 0$ for $t \in [s_1, s_2]$. Suppose that $\dot{x}_i$ changes sign in $[s_1, s_2]$ so there is a $\tau \in [s_1, s_2]$ so that for $s_1 \leq t < \tau$, $\dot{x}_i(t) < 0$ and $\dot{x}_i(\tau) = 0$. Then

$$\ddot{x}_i(\tau) = \frac{d}{dt}\dot{x}_i(\tau) = \frac{d}{dt}n(x_{i-1}(\tau) - x_i(\tau)) = n\dot{x}_{i-1}(\tau) < 0$$

So τ must be a local maximum, but this contradicts the fact that for $s_1 \leq t < \tau$, $\dot{x}_i(t) < 0$. ■

Proof of Lemma 7.12. Without loss of generality, assume that $y_i > 0$ just to the left of z_1 and just to the right of z_2 and $y_i < 0$ in (z_1, z_2). Let ϵ be small enough that y_{i-1} has no zero in $[z_1 - \epsilon, z_1 + \epsilon]$ (if $\epsilon = 0$ then we are done). Then we must have $y_{i-1} < 0$ in $[z_1 - \epsilon, z_1]$, since otherwise y_i would change sign before y_{i-1} and so Lemma 7.11 applies with $s_1 = z_1 + \epsilon$. Hence y_{i-1} must change sign before z_2 since that is where y_i changes sign. ■

Proof of Lemma 7.13. This is a corollary of Lemma 7.12.

We conclude by proving that solutions of (7.1) satisfy **A5**. We have the following Lemma.

Lemma 7.14. *Suppose x is a solution of (7.1) with $N(x_t) = k$ for all $t \in [t_1, \infty)$. Then between any two zeroes of $\dot{x}$ in the interval $[t_1+2, \infty)$ there is a zero of x. The same result holds if we replace the interval $[t_1, \infty)$ by $(-\infty, t_1]$ and $[t_1 + 2, \infty)$ with $(-\infty, t_1 - 2]$.*

Proof. Suppose this is not true and that there are consecutive zeroes, b_1, $b_2 \in [t_1+2, \infty)$, of $\dot{x}$ so that x has no zeroes in $[b_1, b_2]$. First notice that $b_2 - b_1 < 1$ since if $\dot{x}(b_2) = 0$ then $x(b_2-1) = 0$. So we can choose $c > t_1+1$ so that $[b_1, b_2] \subset [c, c+1]$ and $x(c) \neq 0 \neq x(c+1)$.

Let ℓ be the number of zeroes of x in $[c, c+1]$. Call these zeroes $z_1, ..., z_\ell$. Let m be the number of zeroes of $\dot{x}$ in $[c, c+1]$. Since there must be at least one zero of $\dot{x}$ in every interval $[z_i, z_{i+1}]$ (since f is continuous and x is continuous), we know that

$$m \geq \ell - 1 \tag{7.4}$$

We will prove that in fact $m = \ell + 1$ and that this implies $\dot{x}(c)x(c-1) > 0$ which contradicts the negative feedback condition. We will consider separately the three cases: $[b_1, b_2] \subset [z_i, z_{i+1}]$, $[b_1, b_2] \subset [c, z_1]$ and $[b_1, b_2] \subset [z_\ell, c+1]$.

Suppose $[b_1, b_2] \subset [z_i, z_{i+1}]$. In this case there must be another zero of $\dot{x}$ in $[z_i, z_{i+1}]$ and so, by (7.4), in fact

$$m \geq \ell + 1 \tag{7.5}$$

Since the number of zeroes in x in $[c-1, c]$ is m, either $N(x_c) = m$ or $N(x_c) = m+1$. Either way, $N(x_c) \geq \ell + 1$. Since $N(x_t)$ is constant for $t \geq t_1$, then also $N(x_{c+1}) \geq \ell + 1$. But since there are ℓ zeroes of x in $[c, c+1]$ we must have $N(x_{c+1}) = \ell + 1$ and $x(c)x(c+1) > 0$. Also we conclude that $N(x_c) = \ell + 1$ and so, by (7.5), we have

$$m = \ell + 1. \tag{7.6}$$

Since there are m zeroes i $(c-1, c]$ and $N(x_c) = m$, we must have

$$x(c-1)x(c) < 0. \tag{7.7}$$

Since there are already at least three zeroes of $\dot{x}$ in $[z_i, z_{i+1}]$ and at least one zero of $\dot{x}$ in each other interval $[z_j, z_{j+1}]$, we must have in fact that there are exactly three zeroes of $\dot{x}$ in $[z_i, z_{i+1}]$, exactly one zero in each other interval $[z_j, z_{j+1}]$ and no zeroes of $\dot{x}$ in either $[c, z_1]$ or $[z_\ell, c+1]$. Since there are no zeroes in $[c, z_1]$, we must have $\dot{x}(c)x(c) < 0$ and so this together with (7.7) implies that

$$\dot{x}(c)x(c-1) > 0 \tag{7.8}$$

Now suppose that $[b_1, b_2] \subset [c, z_1]$. Since every interval $[z_i, z_{i+1}]$ must have at least one zero of $\dot{x}$, we again conclude that (7.5) holds. Since (7.6) and (7.7) are results only of the assumptions and (7.5), they also hold. Then $\dot{x}$ must have exactly two zeroes in $[c, z_1]$, one zero in each interval $[z_i, z_{i+1}]$ and no zeroes in $[z_\ell, c+1]$. Since there are two zeroes in $[c, z_1]$, again we must have $\dot{x}(c)x(c) < 0$ and again we conclude (7.8).

If $[b_1, b_2] \subset [z_\ell, c+1]$, then the proof goes as in the previous paragraph except that in this case we would have no zeroes of $\dot{x}$ in $[c, z_1]$ and two in $[z_\ell, c+1]$. Since there are no zeroes of $\dot{x}$ in $[c, z_1]$, we again conclude (7.8).

Since (7.8) contradicts the negative feedback condition, we are done. ■

¿From Proposition 6.6d, we know that $V(x_t) = N(x_t)$ for $t \in [t_5, \infty)$, so the hypotheses of Lemma 7.14 are satisfied and the assumption **A5** holds for (7.1).

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