Upper Semicontinuity of Morse Sets of a Discretization of a Delay-Differential Equation:A Complete Solution

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In this paper, we consider a discrete delay problem with negative feedback $\dot{x}(t) = f(x(t), x(t-1))$ along with a certain family of time discretizations with stepsize $1/n$. In the original problem, the attractor admits a Morse decomposition. We proved in [G,H] that the discretized problems have global attractors. It was proved in [G,M] that such attractors also admit Morse decompositions. In [G,H] we proved certain continuity results about the individual Morse sets, including that if $f(x,y) = f(y)$, then the individual Morse sets are upper semicontinuous at $n = \infty$. In this paper we extend this result to the general case; that is we prove for general $f(x,y)$ with negative feedback that the Morse sets are upper semicontinuous.

I. Introduction

In a previous paper, [G,H], we considered the relationship between Morse sets (when they existed) for the following problems.

$$\begin{aligned} \dot{x}(t) &= f(x(t), x(t-1)) \\ x(t) &= \phi(t), \quad t \in [-1, 0] \end{aligned} \qquad (1.1)_\infty$$

[1] Research partially supported by NSF grant 291222

and

$$\begin{aligned}
\dot{y}_0(t) &= f(y_0, y_n) \\
\dot{y}_1(t) &= n(y_0 - y_1) \\
&\vdots \\
\dot{y}_n(t) &= n(y_{n-1} - y_n) \\
\\
y_0(0) &= \phi(0) \\
y_1(0) &= \phi(-\frac{1}{n}) \\
&\vdots \\
y_n(0) &= \phi(-1)
\end{aligned} \tag{1.1$_n$}$$

Problem $(1.1)_\infty$ is a discrete delay problem with negative feedback and $(1.1)_n$ is a time discretization of $(1.1)_\infty$ which we choose so that $\dot{y}_k(0)$ is the slope of the secant line from $\phi(-k/n)$ to $\phi(-(k-1)/n)$.

Basically, our previous result ([G,H]) says that if $\{S_k^n\}_{k=1}^{N_n}$ are Morse sets for $(1.1)_n$ and $\{S_k^n\}_{k=1}^{N_\infty}$ are Morse sets for $(1.1)_\infty$, then, under assumptions **A1** through **A3** below, for any $\epsilon > 0$, there exists N so that for all $n > N$, S_k^n is in an ϵ-neighborhood of

$$M_k := \left(\cup_{j \le k} S_j^\infty\right) \cup \left(\cup_{j,l \le k} C^\infty(j,l)\right)$$

for all $1 \le k \le N_\infty$. $C^\infty(j,l)$ is the set of all connecting orbits with α-limit set in S_j^∞ and ω-limit set in S_l^∞.

In the case that $(1.1)_\infty$ has the form

$$\dot{x}(t) = f(x(t-1))$$

we proved the stronger result that, under assumptions **A1** through **A3** below, for any $\epsilon > 0$, there exists N so that for all $n > N$, S_k^n is in an ϵ-neighborhood of S_k^∞ for all $1 \le k < N_\infty$

In this paper, we prove that the stronger result indeed holds for the more general problem $(1.1)_\infty$ without any further assumptions.

In the remainder of the introduction, we state the required assumptions and define the Morse decompositions. We then give a more precise statement of our earlier results and we conclude with a statement of our new result.

First we require certain assumptions on f, which we will refer to collectively as assumption (**A1**).

A1a. $f : \mathbb{R}^2 \to \mathbb{R}$ is C^∞
A1b. $\eta f(0, \eta) < 0$ for all $\eta \neq 0$
A1c. $A + B < 0$ and $B < 0$ where $A = \partial f(\xi, \eta)/\partial \eta|_{(0,0)}$ and $B = \partial f(\xi, \eta)/\partial \xi|_{(0,0)}$

We will also assume that $(1.1)_\infty$ admits a global attractor. To state this assumption precisely, we must specify the function space in which we usually consider $(1.1)_\infty$ and define the flow in that space. Choose an initial condition $\phi \in C := C([-1, 0], \mathbb{R})$ and let $x(t)$ be the solution with $x(\theta) = \phi(\theta)$ for $\theta \in [-1, 0]$. We can define a solution of $(1.1)_\infty$ as an element in C by defining the function $x_t \in C$ as $x_t(\theta) = x(t + \theta)$ for $\theta \in [-1, 0]$. We then define the solution operator $T_\infty(t)\phi = x_t$. The collection $\{T_\infty(t)\}_{t\geq 0}$ is a semigroup and the action of this semigroup on C defines a semiflow. We denote the set of all bounded solutions of $(1.1)_\infty$ as $\hat{A}_\infty \subset C((-\infty, \infty), \mathbb{R})$ and define $\mathcal{A}_\infty \subset C$ as the set of all initial conditions which give rise to a solution in $\hat{A}_\infty$. The semiflow given by $\{T_\infty(t)\}_{t\geq 0}$ can be extended to a flow $\{T_\infty(t)\}_{t\in \mathbb{R}}$ on $\mathcal{A}_\infty$. We then assume that

A2. $\mathcal{A}_\infty$ is a global attractor.

It was proved in [G,H], using results from [H], that if $(1.1)_\infty$ admits a global attractor, then so does $(1.1)_n$ for large n and the attractors are upper semicontinuous with respect to the parameter n. It was proved in [G,M] that a global attractor for $(1.1)_n$ must have a Morse decomposition. Since the Morse decomposition will be given in $\hat{A}_\infty$, we must define the flow on $\hat{A}_\infty$. The flow will just be translation by time. If $\hat{x} \in \hat{A}_\infty$, then $\phi := \hat{x}|_{[-1,0]} \in \mathcal{A}_\infty$ and $\hat{x}$ is the solution through initial condition ϕ. For $\theta \in (-\infty, \infty)$, define $\hat{x}_t(\theta) := \hat{x}(t + \theta)$.

In [M-P], the author defined a discrete Lyapunov function on $\hat{A}_\infty$, $V : C((-\infty, \infty), \mathbb{R}) \to \mathbb{N}$. Define $\sigma := \inf\{t \geq 0 : \hat{x}(t) = 0\}$ if it exists. Then, if σ exists, define $V(\hat{x})$ to be the number of zeroes, counting multiplicity, of $\hat{x}$ in the interval $(\sigma - 1, \sigma]$. Otherwise, define $V(\hat{x}) = 1$. The author then proved that, in $\hat{A}_\infty$, V is bounded above, takes odd integer values and $V(\hat{x}_t)$ is nonincreasing in t. It is also proved in [M-P] that under assumptions **A1** and **A2** $\hat{A}_\infty$ admits a Morse decomposition which we now describe.

The Morse sets will be disjoint, compact invariant sets in $\hat{A}_\infty$ on which the Lyapunov function is constant. The Morse sets are ordered $S_1^\infty < S_2^\infty < \ldots < S_M^\infty$ so that for any $\phi \in \hat{A}_\infty$, there are positive integers j and k, $j \leq k$ such that $\alpha(\phi) \subset S_k^\infty$ and $\omega(\phi) \subset S_j^\infty$. One of the Morse sets will be the set $\{0\}$. The number of Morse sets which are below $\{0\}$ in the ordering depends on the number of eigenvalues of the linearization at zero which have positive real part. Specifically, the characteristic equation obtained by setting $x = e^{\lambda t}$ in the linearization of $(1.1)_\infty$ is

$$-\lambda + A + Be^{-\lambda} = 0. \tag{1.2}$$

We assume

A3 The zero solution of $(1.1)_\infty$ is hyperbolic; that is, there are no roots of (1.2) with zero real part.

Then, if N is the number of roots of (1.2) with positive real part, there are $N/2$ Morse sets below $\{0\}$ (it is proved in [M-P] that N is even). Let $N^* := N/2+1$. The sets are defined as follows. For $1 \leq k \leq N^*-1$ and $N^*+1 \leq k$

$$S_k := \{\hat{x} \in \hat{A}_\infty \setminus \{0\} : V(x_t) = 2k-1 \text{ for all } t \text{ and } 0 \notin \alpha(\hat{x}) \cup \omega(\hat{x})\}$$

and

$$S_{N^*} := \{0\}.$$

It is proved in [M-P] that the function V is bounded on the attractor $\hat{A}_\infty$ and that the number of nonempty sets S_k is finite, say M, and that the sets $\{S_k\}_{1\leq k\leq M}$ form a Morse decomposition of $\hat{A}_\infty$.

To describe the Morse decomposition for $(1.1)_n$, we begin by defining operators analogous to T_∞. If $\mathbf{y_0}$ is an initial condition in $I\!R^{n+1}$ then $\tilde{T}_n(t)\mathbf{y_0} \in I\!R^{n+1}$ will be the solution through $\mathbf{y_0}$ at time t. We will eventually drop the tilde notation when we have chosen a function space in which we can compare solutions of $(1.1)_n$ with solutions of $(1.1)_\infty$.

Systems of the form in $(1.1)_n$ are commonly known as *cyclic feedback systems* and have been studied by Mallet-Paret and Smith [M-P,S], Gedeon and Mischaikow [G,M], and Gedeon[G]. In fact, assumption **A1**b guarantees that this is a *negative cyclic feedback system.* In [M-P,S], Mallet-Paret and Smith define a discrete Lyapunov function for $(1.1)_n$, in the case that it admits a global attractor $\tilde{\mathcal{A}}_n$. For $1 \leq i \leq n$, define $\delta_i = 1$ and define $\delta_0 = -1$. For a vector $\langle x_0, ... x_n\rangle \in I\!R^{n+1}$ with $x_i \neq 0$, define

$$\tilde{V}_n(\langle x_0, x_1, ..., x_n\rangle) = \text{ card}\{i : \delta_i x_i x_{i-1} < 0\}$$

where we define $x_{-1} = x_n$. $\tilde{V}_n$ counts the number of sign changes in the vector and adds one if the first and last element have the same sign. We extend $\tilde{V}_n$ by continuity whenever possible. If the vector $\mathbf{x}(t) = \langle x_0(t), x_1(t), ..., x_n(t)\rangle$ is a solution of $(1.1)_n$, then $\tilde{V}_n(\mathbf{x}(t))$ is nonincreasing. More precisely, if $\mathbf{x}(t)$ is in a region where $\tilde{V}_n$ is defined, then $\tilde{V}_n(\mathbf{x}(t))$ is constant. If $\tilde{V}_n$ is not defined at $\mathbf{x}(t)$, then for small ϵ, $\tilde{V}_n(\mathbf{x}(t-\epsilon)) = 2 + \tilde{V}_n(\mathbf{x}(t+\epsilon))$ ($\tilde{V}_n(\mathbf{x}(t+\epsilon)) < \tilde{V}_n(\mathbf{x}(t-\epsilon))$) . Clearly, $\tilde{V}_n$ is bounded and takes odd integer values. We will use this Lyapunov function to define the Morse sets. Again the number of Morse sets depends on the number of eigenvalues with positive real part. In [G,H] we prove that if **A3** holds then the zero solution of the linearization of $(1.1)_n$ is hyperbolic.

Let K_n be the number of eigenvalues with positive real part. If K_n is even, define $K_n^* := K_n/2+1$; if K_n is odd, define $K_n^* := (K_n+1)/2+1$. There are K_n^*-1 Morse sets below $\{0\}$. Suppose there are J_n Morse sets all together. They are defined as follows. For $1 \leq k \leq K_n^*-1$ or $K_n^* \leq k \leq J_n$

$$\tilde{S}_k^n := \{\mathbf{x} \in \tilde{\mathcal{A}}_n : \tilde{V}_n(\tilde{T}_n(t)\mathbf{x}) = 2k-1 \text{ for all } t, 0 \notin \alpha(\mathbf{x}) \cup \omega(\mathbf{x})\})$$

and

$$\tilde{S}_{K^*}^n := \{0\}$$

It was proved in [G,M] that a coarser decomposition, where $\tilde{S}^n_{K^*} := \{0\} \cup \bigcup_{k>K^*} \tilde{S}^n_k$, is a Morse decomposition of $\tilde{\mathcal{A}}_n$. The same argument also proves that $\{\tilde{S}^n_k\}_{k=1}^{J_n}$ is a Morse decomposition of $\tilde{\mathcal{A}}_n$.

Hence for each problem $(1.1)_n$, $n \leq \infty$, we have an attractor and a Morse decomposition. These Morse decompositions are not unrelated. But it is not clear how we can compare these two problems. It turns out that both the infinite dimensional problem and the finite dimensional are connected to the following distributed delay problem.

$$\begin{aligned} \dot{x}(t) &= f\left(x(t), \int_{-\infty}^{0} x(t+s)Q_n(s)ds\right) \\ x_0 &= \phi, \quad \phi \in C((-\infty, 0]) \end{aligned} \tag{1.3$_n$}$$

where

$$Q_n(s) = n^n \frac{(-s)^{n-1}}{(n-1)!} e^{ns}$$

With this kernel, we obtain the system in $(1.1)_n$ if we make the following change of variables (see [B,T])

$$\begin{aligned} y_0(t) &= x(t) \\ y_k(t) &= \int_{-\infty}^{0} x(t+s) r^n_k(s)ds \end{aligned}$$

where

$$r^n_k(s) = n^k \frac{(-s)^{k-1}}{(k-1)!} e^{ns}$$

The initial conditions will be

$$\begin{aligned} y_0(0) &= \phi(0) \\ y_k(0) &= \int_{-\infty}^{0} \phi(s) r^n_k(s)ds \end{aligned} \tag{1.4}$$

Problem $(1.1)_\infty$ is the "limit" of the problems $(1.3)_n$ in the sense that the kernels Q_k converge weakly to the δ-function at -1; that is, for bounded functions x (in fact, for functions in the space X defined below), we have

$$\int_{-\infty}^{0} x(s)Q_n(s)ds \to x(-1) \quad \text{as} \quad n \to \infty.$$

The convergence of the kernels allows us to make use of results in [H] about the dependence of attractors on the delay. To state these results, we must first give a function space in which we can compare solutions for different values of n. The choice of function space is discussed extensively in [H]. We choose the space

$$X := \{\phi : (-\infty, 0] \to I\!R \,|\, \phi \text{ is continuous on } [-2, 0] \text{ and } \|\phi\|_X < \infty\}$$

where

$$\|\phi\|_X := \sup_{-2\leq s\leq 0} |\phi(s)| + \int_{-\infty}^0 |\phi(s)|Q_1(s)ds.$$

It can be shown that the problems $(1.3)_n$ and $(1.1)_\infty$ are well-defined in X. The attractor for $(1.1)_\infty$ in X is just the backward flow through all elements in the attractor in C. This construction is discussed completely in [H]. We will call the attractor in X $\mathcal{A}_\infty$ also. We define solution operators $T_n(t)$ for $(1.3)_n$ in X, $1 \leq n \leq \infty$ analogous to the operators $\tilde{T}_n(t)$. We can give the Morse decomposition for $(1.3)_n$ in terms of $(1.1)_n$. The Morse decomposition in $\tilde{\mathcal{A}}_n$ gives rise to a Morse decomposition in $\mathcal{A}_n$. Define

$$S_k^n := \{\phi \in \mathcal{A}_n | \langle \phi(0), \int_{-\infty}^0 \phi(s) r_1^n(s)ds, \int_{-\infty}^0 \phi(s) r_2^n(s)ds, ..., \int_{-\infty}^0 \phi(s) r_n^n(s)ds \rangle \in \tilde{S}_k^n\}$$

We proved in [G,H] that this indeed gives a Morse decomposition of $\mathcal{A}_n$.

Now we state precisely the results which were proved in [G,H].

Theorem 0.1. *Assume that assumptions* **A1** *through* **A3** *are satisfied. Then for any $\epsilon > 0$ there exists N so that for all $n > N$, the following hold.*

a. *$N^* = K_n^*$; that is, the number of Morse sets in the decompositin of $\mathcal{A}_\infty$ which lie below $\{0\}$ is the same as the number of Morse sets in the decomposition of $\mathcal{A}_n$ which lie below $\{0\}$.*
b. *For all $1 \leq k \leq N^*$, S_k^n is in an ϵ-neighborhood of*

$$M_k := \left(\cup_{j\leq k} S_j^\infty\right) \cup \left(\cup_{j,l\leq k} C^\infty(j,l)\right)$$

where $C^\infty(j,l)$ is the set of all connecting orbits with α-limit set in S_j^∞ and ω-limit set in S_l^∞.

If $(1.1)_\infty$ has the form

$$\dot{x}(t) = f(x(t-1))$$

then the theorem becomes

Theorem 0.2. *Assume that assumptions* **A1** *through* **A3** *are satisfied. Then for any $\epsilon > 0$ there exists N so that for all $n > N$, the following hold.*

a. *$N^* = K_n^*$; that is, the number of Morse sets in the decompositin of $\mathcal{A}_\infty$ which lie below $\{0\}$ is the same as the number of Morse sets in the decomposition of $\mathcal{A}_n$ which lie below $\{0\}$.*
b. *For all $1 \leq k < N^*$, S_k^n is in an ϵ-neighborhood of S_k^∞.*

Now we state the result which will be proved in this paper.

Theorem 1.1. *Consider the problem* $(1.1)_n$ *and* $(1.1)_\infty$ *where the right-hand side is allowed to have the general form* $f(x(t), x(t-1))$. *Assume that assumptions* **A1** *through* **A3** *are satisfied. Then for any* $\epsilon > 0$, *there exists* N *so that for all* $n > N$, S_k^n *is in an* ϵ*-neighborhood of* S_k^∞ *for all* $1 \leq k \leq M$. *That is, the sets* $\{S_k^n\}$ *are upper semicontinuous at* $n = \infty$.

Before we go on to the proof of Theorem 1.1, we should explain what we mean when we say that $\{S_k^n\}$ is upper semicontinuous in the case that $N^* \leq k \leq M$. We don't know if the problems $(1.1)_n$ and $(1.1)_\infty$ have the same number of Morse sets above $\{0\}$. However, if given k, there are an infinite number of $n > N$, call them n_j, such that $S_k^{n_j}$ is not empty, then S_k^∞ is not empty and the sets $\{S_k^{n_j}\}$ are upper semicontinuous. If there exists N such that for $n > N$ the sets S_k^n are empty, then even if S_k^∞ is not empty, the sets $\{S_k^n\}$ are upper semicontinuous by definition.

II. Some Results From [G,H]

Here we state several central results from [G,H]. The first is given as a theorem in [M-P].

Theorem 2.1a. *Assume that* **A1** *through* **A3** *hold. If* $x \in \hat{A}_\infty$ *then either* $x(t) \to 0$ *as* $t \to \infty$ *or* x *satisfies the following.*

1. $\liminf_{t\to\infty}(|x(t)| + |\dot{x}(t)|) > C$ *where* $C > 0$ *is independent of* x.
2. *There exist* $t_1 > 0$ *such that all zeroes of* x *which lie in* $[t_1, \infty)$ *are simple.*
3. *There exist* $t_2 > 0$ *and* $d > 0$ *such that if* z_1 *and* z_2 *are two zeroes of* x *in* $[t_3, \infty)$, *then we have* $|z_1 - z_2| > d$.

Since the solution operator is a group in $\mathcal{A}_\infty$, we also have

Corollary 2.1b. *Assume that* **A1** *through* **A3** *hold. If* $x \in \hat{A}_\infty$ *then either* $x(t) \to 0$ *as* $t \to -\infty$ *or* x *satisfies the following.*

1. $\liminf_{t\to-\infty}(|x(t)| + |\dot{x}(t)|) > C$ *where* $C > 0$ *is independent of* x
2. *There exist* $t_1 < 0$ *such that all zeroes of* x *which lie in* $(-\infty, t_1]$ *are simple.*
3. *There exist* $t_2 < 0$ *and* $d > 0$ *such that if* z_1 *and* z_2 *are two zeroes of* x *in* $(-\infty, t_3]$, *then we have* $|z_1 - z_2| > d$.

The following Lemma is a Hartman-Grobman type result whose corollary we shall use later in the paper. For $\mathbf{x} \in I\!R$, let $\|\mathbf{x}\|_1 := max_{0 \leq i \leq n} x_i$.

Lemma 2.2 (Lemma 3.2 in [G,H]). *Let us denote the vector field given by* $(1.1)_n$ *by* $F_n(\mathbf{x})$. *There is a constant* $\zeta > 0$ *such that for all* n *there is a homeomorphism* h_n *such that*

$$DF_n(0) \circ h_n(\mathbf{x}) = h_n \circ F_n(\mathbf{x})$$

for all $\|\mathbf{x}\|_1 \leq \zeta$.

Corollary 2.3. *If a solution x^n of $(1.1)_n$ satisfies $|x^n(t)| < \zeta$ for all $t < 0$ then $x^n \in W^u(0)$.*

The remaining results are various convergence results with respect to the parameter n.

Lemma 2.4 (Lemma 4.1 in [G,H]). *Suppose $x^n \in \mathcal{A}_n$ and $x^n \to x \in \mathcal{A}_\infty$ where the convergence is in the X-norm. Then, given ϵ and T, there exists $N(\epsilon, T)$ so that for $n > N$, $|x^n(t) - x(t)| < \epsilon$ for all $t \in [-T, 0]$.*

Lemma 2.5 (Lemma 4.5 in [G,H]). *Suppose $x^n \in \mathcal{A}_n$ and $x^n \to x \in \mathcal{A}_\infty$ in X. Then, given ϵ and T, there exists $N(\epsilon, T)$ so that for $n > N$, $|\dot{x}^n(t) - \dot{x}(t)| < \epsilon$ for all $t \in [-T, 0]$.*

To state the next lemma, we introduce the notation

$$x_j^n(t) = \int_{-\infty}^{0} x^n(t+s) r_j^n(s)\, ds$$

Fix $0 \le l \le 1$ and choose a sequence $k(n)/n \to l$, $0 \le k(n) \le n$.

Lemma 2.6. *Suppose $x^n \in \mathcal{A}_n$ and $x^n \to x \in \mathcal{A}_\infty$ in X. For every $\epsilon > 0$ and $T > 0$ there exists N so that for all $n > N$ and for all $t \in [-T, 0]$,*

$$\left| x(t-l) - \int_{-\infty}^{0} x^n(s+t) r_{k(n)}^n(s)\, ds \right| < \epsilon$$

Proof By Lemma 4.2 in [G,H], the sequence $\{x^n\}$ is bounded and equicontinuous and so satisfies the hypotheses of Lemma 4.3 in [G,H]. The conclusion is then a direct result of Lemma 4.3 ([G,H]). ■

Lemma 2.7. *Suppose $x^n \in \mathcal{A}_n$ and $x^n \to x \in \mathcal{A}_\infty$ in X. Then given $a \in I\!R$ and $\epsilon > 0$ there exists N such that for all $n > N$,*

$$|\dot{x}(a-l) - \dot{x}_{k(n)}^n(a)| < \epsilon.$$

The choice of N is independent of $l \in [0,1]$ and independent of the sequence $k(n)/n$.

Proof Let $v_n = \dot{x}^n$. From Lemma 2.5, we know that there exists $N(\epsilon, a)$ so that for $n > N$, $|\dot{x}^n(t) - \dot{x}(t)| < \epsilon$ for all $t \in [a-2, 0]$. By Lemma 4.2 in [G,H], the sequence $\{v_n\}_0^\infty$ is equicontinuous and equibounded. Hence the sequence $\{v_n\}$ satisfies the hypotheses of Lemma 4.3 ([G,H]) and so

$$|\dot{x}(t-l) - \int_{-\infty}^{0} \dot{x}_n(s+t) r_{k(n)}^n(s) ds| < 2\epsilon$$

for all $t \in [a, 0]$. In particular, this holds for $t = a$. To see that N is independent of $l \in [0, 1]$ and independent of the sequence $k(n)/n$, we must check that the steps in the proof of Lemma 4.3 ([G,H]) are independent. This is indeed the case because of the following facts: $\int_{-\infty}^{0} r_j^n(s)ds = 1$ for all j, n; $r_j^n(-2)$ has a bound which depends only on n, not on j; and the integral of $r_{k(n)}^n$ outside some small interval containing $k(n)/n$ has a bound which depends only on n (for details, see the proof of Lemma 4.3 in [G,H]). ■

III. Counting Sign Changes and Zeroes

Suppose that for an infinite number of n, $x^n \in S_k^n$. It is shown in [G,H] that the sequence $\{x^n\}$ has a convergent subsequence $x^{n_i} \to x$ where $x \in \mathcal{A}_\infty$ and $x(t)$ is a solution of $(1.1)_\infty$ for all t. We will prove that, in fact, $x \in S_k^\infty$. This completes the proof since

Lemma 3.1. *Suppose that every sequence $\{x^n\}$ with $x^n \in S_k^n$ has a convergent subsequence which converges to an element $x \in S_k^\infty$. Then for every ϵ there is an N such that for all $n > N$, $S_k^n \subset N_\epsilon(S_k^\infty)$.*

This was proven in [G,H] (Lemma 6.3) replacing S_k^∞ with M_k defined in the introduction. The proof is the same for S_k^∞. From here on we write x^n for the subsequence.

In order to prove that $x \in S_k^\infty$, we make use of Theorem 2.1a and Corollary 2.1b. Hence we must make sure that the hypotheses of the theorem and corollary hold. We require

(S1) $x(t) \not\to 0$ as $t \to \infty$
(S2) $x(t) \not\to 0$ as $t \to -\infty$.

In [G,H] we proved Theorem 0.1 in the following way. We first proved that if **(S2)** holds then $V(x_t) \leq 2k - 1$ for all t. Then we showed that indeed for $k < N^*$ (recall that N^* is the number of Morse sets below $\{0\}$), **(S2)** holds (see Lemma 6.7 of [G,H]). Here we take a similar approach. We first prove

Proposition 3.2. *If* **(S1)** *holds, then $V(x_t) \geq 2k - 1$ for all t.*

Then we show

Proposition 3.3. *If $k > N^*$ then* **(S1)** *holds.*

Proposition 3.4. *If $k < N^*$ then* **(S1)** *holds.*

To prove Theorem 1.1, we also need to show that the conclusion of Theorem 0.1 holds for the case that $k > N^*$. So we prove the following proposition.

Proposition 3.5. *If $k > N^*$ then* **(S2)** *holds.*

The remainder of this section is organized as follows. In section 3.1, we prove Proposition 3.2. In section 3.2, we prove Propositions 3.3, 3.4 and 3.5. In section 3.3, we conclude the proof of Theorem 1.1.

3.1 The Proof of Proposition 3.2

Since **(S1)** holds, we can apply Theorem 2.1.a. There exists t_1 such that for all $t > t_1$, $|x(t)| + |\dot{x}(t)| > C$. There exists t_2 such that all zeroes of x in (t_2, ∞) are simple. There exists t_3 such that if z_1 and z_2 are two zeroes of x in (t_3, ∞) then $|z_1 - z_2| > d$. Since V is bounded below and above and takes only integer values, there exists t_4 such that $V(x_t)$ is constant for all $t > t_4$. If we define $\bar{\eta}$ so that

$$\bar{\eta} := \min\left(\frac{2dC}{d+4}, \frac{C}{2}\right)$$

then we have the following lemma.

Lemma 3.6. *There is an $a > \max(t_1, t_2, t_3, t_4)$ and an $\eta < \bar{\eta}$ so that $|x(a)| > \eta$ and $|x(a-1)| > \eta$.*

Define the vectors

$$X^n := \langle x^n(a), \int_{-\infty}^{0} x^n(a+s) r_1^n(s) ds, ..., \int_{-\infty}^{0} x^n(a+s) r_n^n(s) ds \rangle$$

and

$$D^n := \langle x^n(a), x^n(a), x^n(a - \frac{1}{n}), x^n(a - \frac{2}{n}), ..., x^n(a - 1 + \frac{1}{n}) \rangle$$

We proved in [G,H] that if n is large enough, D^n catches all the sign changes of x^n; that is

Lemma 3.7 (Proposition 6.6b in [G,H]). *If n is large enough then $\tilde{V}(D^n) = N(T_n(a)x^n)$, where*

$$N(y) := \text{ the number of sign changes of } y \text{ in } (-1, 0] + \frac{sgn\ (y(0)y(-1)) + 1}{2}$$

Here we assume that n is large enough to satisfy Lemma 3.7. From Section 2 (Lemmas 2.4 and 2.5) we know that we can pick n large enough so that $|x^n(t)| + |\dot{x}^n(t)| > C/2$ for all $t \in [a-1, a]$. Hence in any region of $[a-1, a]$ where $|x^n(t)| < C/4$, we must have $|\dot{x}^n(t)| > C/4$ so $x^n(t)$ must be monotone in such a region. Let $[b_1, b_2] \subset [a-1, a]$ be such that $|x^n(b_1)| = C/4$ and $|x^n(b_2)| = C/4$ and x^n is monotone in $[b_1, b_2]$, so in the interval $[b_1, b_2]$, $|x^n(t)| < C/4$ and $|\dot{x}^n(t)| \neq 0$. Without loss of generality assume $\dot{x}^n(t) > 0$ for $t \in [b_1, b_2]$. Given n, there exists $m = m(n)$ and $s = s(n)$ such that $a - j/n \in [b_1, b_2]$ for all $m \leq j \leq s$ and $a - j/n \notin [b_1, b_2]$ if $j < m$ or $j > s$. For these $s - m + 1$ consecutive elements of the vector D^n we have

$$x^n(a - j/n) > x^n(a - (j+1)/n)$$

for all $j = m, \ldots, s-1$. Then we have the following Lemma.

Lemma 3.8. *For large enough n, if $x^n(a - j/n) > x^n(a - (j + 1)/n)$, for all $j = m(n), \ldots, s(n) - 1$, then also*

$$x_j^n(a) > x_{j+1}^n(a)$$

for all $j = m(n), \ldots, s(n) - 1$.

Proof Suppose $a - l \in [b_1, b_2]$ and $k(n)/n \to l$, $0 \leq k(n) \leq n$. By Lemma 2.5, for all $\epsilon > 0$ there exists N such that for all $n > N$,

$$|\dot{x}(a - l) - \dot{x}_{k(n)}^n(a)| < \epsilon$$

N can be chosen independently of $l \in [0, 1]$ and independently of the seqeunce $\frac{k(n)}{n}$. Let $L = \dot{x}(a - l) > 0$ and $\epsilon = L/2$. Then there exists N such that for all $n > N$ we have

$$|L - \dot{x}_{k(n)}^n(a)| < \frac{L}{2}$$

so $\dot{x}_{k(n)}^n(a) > 0$. But

$$\dot{x}_{k(n)}^n(a) = n(x_{k(n)-1}^n(a) - x_{k(n)}^n(a)) > 0$$

so

$$x_{k(n)-1}^n(a) > x_{k(n)}^n(a).$$

This proves the result since for any $m(n) \leq j \leq s(n) - 1$, there exists an l and a sequence $k(n)$ so that for some n, $k(n)/n = (j + 1)/n$. ∎

¿From Lemma 3.8, we can conclude that the mth through $(s-1)$st terms in the vector X^n are monotone increasing. The analogous result holds if $x^n(a-j/n) < x^n(a-(j+1)/n)$ for all $j = m, \ldots, s - 1$. We use this to prove

Lemma 3.9. *For n large enough $\tilde{V}_n(D^n) \geq \tilde{V}_n(X^n)$.*

Proof By Lemma 2.5, we can pick n large enough so that

$$|x^n(a - j/n) - x_j^n(a)| < \frac{\eta}{2} \tag{3.1}$$

for all $j = 1, \ldots, n$. Let

$$\begin{aligned}
b_1 &= \{\inf t : \ t > a - 1 \text{ and } |x(t)| = C/4\} \\
b_2 &= \{\inf t : \ t > b_1 \text{ and } |x(t)| = C/4\} \\
b_3 &= \{\inf t : \ t > b_2 \text{ and } |x(t)| = C/4\} \\
&\ \vdots \\
b_q &= \{\sup t : \ t < a \text{ and } |x(t)| = C/4\}
\end{aligned}$$

Then $[a-1,a] = [a-1,b_1] \cup [b_1,b_2] \cup \ldots \cup [b_{q-1},b_q] \cup [b_q,a]$. In an interval $[b_i,b_{i+1}]$, either $|x^n(t)| \geq C/4$ for all $t \in [b_i,b_{i+1}]$ or $|x^n(t)| \leq C/4$ for all $t \in [b_i,b_{i+1}]$ because the derivative $\dot{x}^n(t)$ can only change sign if $|x^n(t)| > C/2$. If for $t \in [b_i,b_{i+1}]$ we have that $|x^n(t)| \geq C/4$ then for j such that $a - j/n \in [b_i,b_{i+1}]$, $x^n_j(a) \neq 0$ by (3.1). If for $t \in [b_i,b_{i+1}]$ we have that $|x^n(t)| \leq C/4$ then let m and s be such that for all $m \leq j \leq s$, $a - j/n \in [b_i,b_{i+1}]$ and for $j < m$ and $j > s$ $a - j/n \notin [b_i,b_{i+1}]$. Then D^n is monotone between its mth and sth elements and, by Lemma 3.8, so is X^n for sufficiently large n. Hence X^n has no more than one sign change between the mth and sth elements (it may have no sign changes if the mth and sth elements of D^n are strictly less than $C/4$ in magnitude).

We consider the far right and left intervals separately. We know that we can pick n large enough so that for all $n > N$ $|x^n(a-1)| > \eta/2$. There are two possiblities for x^n in $[a-1,b_1]$. Either $\eta/2 < |x^n(t)| < C/4$ in $[a-1,b_1]$ or $|x^n(t)| > C/4$ in $[a-1,b_1]$. In either case, we can't change the sign of the vector elements of D^n by adding a perturbation of size less than $\eta/2$. Since we can consider the vector X^n as a perturbation of the vector D^n of size less than $\eta/2$, we can conclude that x^n_j for $a - j/n \in [a-1,b_1]$ all have the same sign. A similar argument holds for the interval $[b_q,a]$.

Hence $\tilde{V}_n(D^n) \geq \tilde{V}_n(X^n)$, for n large enough. ■

Remark 3.10. *With a slightly more careful proof, we can prove in this same way that, in fact,* $\tilde{V}_n(D^n) = \tilde{V}_n(X^n)$*, but Lemma 3.9 is enough for our purposes.*

Now we can conclude the proof of Proposition 3.2. We proved in [G,H] that $V(x_a) = \tilde{V}_n(D^n)$. By definition, we know that $\tilde{V}_n(X^n) = 2k-1$. So by Lemma 3.9 $V(x_a) \geq 2k-1$. Since $V(x_t)$ is decreasing along solutions, $V(x_t) \geq 2k-1$ for all $t \leq a$. Since $a > t_4$, $V(x_t)$ is constant for all $t \geq a$ and we can conclude that $V(x_t) \geq 2k-1$ for all t.

3.2 The Proofs of Propositions 3.3, 3.4 and 3.5

We require the following Lemmas.

Lemma 3.11. *Suppose $x \in \mathcal{A}_\infty$ and $x(t) \to 0$ as $t \to \infty$. Then $x \in W^s(0) \subset X$.*

Proof Since $x \in \mathcal{A}_\infty$, it is bounded in X by a constant which we will call M. So

$$|x(t)| \leq \sup_{-2 \leq s \leq 0} |x_t(s)| + \int_{-\infty}^{0} |x_t(s)| Q_1(s)\, ds = \|x_t\| \leq M$$

Since $x(t) \to 0$ as $t \to \infty$, then, given $\epsilon > 0$, there exists a $\tau_1 > 0$ such that for all $t > \tau_1$, $|x(t)| < \epsilon/3$. By the nature of the kernel Q_1, we can pick τ_2 so that

$$\int_{-\infty}^{-\tau_2} Q_1(s)\, ds < \frac{\epsilon}{3M}$$

Pick $\tau > \tau_1 + \tau_2$. Then for all $t > \tau$ we have

$$\begin{aligned}\|x_t\| &= \sup_{-2\le s\le 0} |x_t(s)| + \int_{-\infty}^{0} |x_t(s)|Q_1(s)\,ds \\ &\le \frac{\epsilon}{3} + \int_{-\infty}^{-\tau_2} |x_t(s)|Q_1(s)\,ds + \int_{-\tau_2}^{0} |x_t(s)|Q_1(s)\,ds \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \int_{-\tau_2}^{0} |x_t(s)|Q_1(s)\,ds\end{aligned}$$

In the last integral above, since $s > -\tau_2$, $t+s > t-\tau_2 > \tau_1+\tau_2-\tau_2 = \tau_1$, so for all $s \in [-\tau_2, 0]$, $|x(t+s)| < \frac{\epsilon}{3}$. So, since $\int_{-\infty}^{0} Q_1(s)\,ds = 1$, we have

$$\|x_t\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

Hence $\|x_t\| \to 0$ as $t \to \infty$ and so $x \in W^s(0)$. ■

Similarly we can show

Lemma 3.12. *Suppose $x \in \mathcal{A}_\infty$ and $x(t) \to 0$ as $t \to -\infty$. Then $x \in W^u(0) \subset X$.*

Proof of Proposition 3.3 Suppose $x^n \to x$ in X and $|x(t)| \to 0$ as $t \to \infty$. Recall that by assumption $x^n \in S_k^n$ for some $k > N^*$ and thus $\tilde{V}(x^n) > 2N^* - 1$. Define $L > 0$ so that for all $t \in [L, -\infty)$, $|x(t)| < \zeta/4$, where ζ is defined in Lemma 2.2. Define

$$q_n = \inf\{t > L : |x^n(t)| = \zeta/2\}$$

We want to show that there is a subsequence $q_{n_i} \to \infty$. Suppose instead that there is a $Q < \infty$ such that $q_n \le Q$ for all n. Let $\tilde{q} > Q$. Pick k so that $L + 2(k-1) < \tilde{q} < L + 2k$. Consider $T_n(L)x^n, T_n(L+2)x^n, \ldots, T_n(L+2k)x^n$. For each $j = 1, \ldots, k$ there exists N_j so that for all $n > N_j$

$$\|T_n(L+2j)x^n - T(L+2j)x\|_X < \zeta/4$$

and so

$$|[T_n(L+2j)x^n](\theta) - [T(L+2j)x](\theta)| < \zeta/4$$

for all $\theta \in [-2, 0]$ and $j = 0, \ldots, k$. Hence for each j, if $n > N_j$,

$$|[T_n(L-2j)x^n](\theta)| < \zeta/2$$

for all $\theta \in [-2, 0]$. Choose $N = \max_j N_j$. Then for all $n > N$, we have

$$|x^n(t)| < \zeta/2$$

for $t \in [L, L+2k]$. Hence for $n > N$, $q_n > \tilde{q}$ and we have reached a contradiction. So such a subsequence exists. For convenience, we also call the subsequence $\{q_n\}$.

Now define

$$y^n(t) = x^n(t + q_n)$$

for all t. Since S_k^n is invariant, $y^n \in S_k^n$ for all n. There is a subsequence, which we again call y^n, so that $y^n \to y$ in X. Then $y \in \mathcal{A}_\infty$. We want to show that $|y(t)| < \zeta$ for all $t < 0$. Suppose there is a $\tilde{t} < 0$ so that $|y(\tilde{t})| = \zeta$. Consider $T(\tilde{t})y$. $T_n(\tilde{t})y_n \to T(\tilde{t})y$ in X so there exists an $\tilde{N}$ so that for all $n > \tilde{N}$,

$$\|T_n(\tilde{t})y_n - T(\tilde{t})y\|_X < \zeta/2$$

and so

$$|[T_n(\tilde{t})y_n](0) - [T(\tilde{t})y](0)| < \zeta/2$$

Pick $N \geq \tilde{N}$ so that for all $n > N$, $L - q_n > \tilde{t}$. Notice that $L = L - q_n + q_n \leq \tilde{t} + q_n < q_n$ so for all $n > N$, $|x^n(\tilde{t} + q_n)| = |[T_n(\tilde{t})y_n](0)| < \zeta/2$ and so

$$|y(\tilde{t})| = |[T(\tilde{t})y](0)| < \zeta$$

and we have arrived at a contradiction. Hence $|y(t)| < \zeta$ for all $t > 0$.

Therefore, by Corollary 2.3, $y \in W^u(0)$. If also $y \in W^s(0)$, then y would be a homoclinic orbit, but according to [M-P] this is impossible. Hence, by Lemma 3.11, $y(t) \not\to 0$ as $t \to \infty$. Hence for y, **(S1)** holds. By Proposition 3.2, $N(y) \geq 2k - 1 > 2N^* - 1$. But since $y \in W^u(0)$, we must have $N(y) \leq 2N^* - 1$ and so we have reached our final contradiction. This proves the Lemma. ■

Proof of Proposition 3.4 If $x(t) \to 0$ as $t \to \infty$, we have shown in Lemma 3.11 that $x \in W^s(0)$, but in that case we must have $V(x) \geq 2N^* - 1$ and hence $k > N^*$. ■

Proof of Proposition 3.5. If $x(t) \to 0$ as $t \to -\infty$, we have shown in Lemma 3.12 that $x \in W^u(0)$, but in that case we must have $V(x) \leq 2N^* - 1$ and hence $k < N^*$. ■

3.3 Completion of the Proof of Theorem 1.1

To prove that $x \in S_k^\infty$ it only remains to show that $0 \notin \omega(x) \cup \alpha(x)$.

Lemma 3.12. $0 \notin \omega(x) \cup \alpha(x)$

Proof We'll prove that $0 \notin \omega(x)$. The proof that $0 \notin \alpha(x)$ follows similarly.

Suppose $0 \in \omega(x)$. Then there exists a sequence $t_j \to \infty$ as $j \to \infty$ such that $\|T_\infty(t_j)x\| \to 0$ as $j \to \infty$. Then for every $\epsilon > 0$ there exists J such that for all $j > J$ we have

$$\|T_\infty(t_j)x\| = \sup_{s\in[-2,0]} |x(t_j + s)| + \int_{-\infty}^{0} |x(t_j + s)|Q_1(s)\ ds < \epsilon$$

so

$$\sup_{s\in[-2,0]} |x(t_j+s)| < \epsilon.$$

Let $\epsilon < C/2$. Then for some j with $t_j > \tau + 2$, where τ is as in the proof of Lemma 3.2, we have $|x(s)| < \epsilon$ for all $s \in [t_j - 2, t_j]$. In this interval, we cannot have $\dot{x}(s) = 0$ since by Lemma 2.1a we would then have $|x(s)| > C$, so in this interval x must be monotone, hence there can be at most one zero of x in $[t_j - 2, t_j]$. This, along with the fact that $|x(s)| < \epsilon$ in $[t_j - 2, t_j]$, implies that the maximum of $|\dot{x}|$ is $2\epsilon/2 = \epsilon$, so in $[t_j - 2, t_j]$ we have $|x(t)| + |\dot{x}(t)| < \epsilon + \epsilon < C$ and this contradicts Lemma 2.1a. ∎

Hence we conclude that $x \in S_k^\infty$ and Theorem 1.1 holds.

References

[B,T] S. N. Busenberg and C.C Travis, Approximation of Functional Differential Equations by Differential Systems, in "Volterra and Functional Differential Equations" (Blacksburg, VA, 1981), 197-205, Dekker, 1982.

[G] T. Gedeon, Cyclic Feedback Systems, *Memoirs of AMS*, vol. 134, No. **637**, (1998).

[G,H] T. Gedeon and G. Hines. Upper Semicontinuity of Morse Sets of a Discretization of a Delay-Differential Equation, *J. Differential Equations*, **151** (1999), 36-78.

[G,M] T. Gedeon and K. Mischaikow, Structure of the Global Attractor of Cyclic Feedback Systems, *J. Dynam. Differential Equations*, **7** (1995), 141-190.

[H] G. Hines, Upper Semicontinuity of the Attractor with Respect to Parameter Dependent Delays, *J. Differential Equations*, **123**(1995), 56-92.

[M-P] J. Mallet-Paret, Morse Decomposition for Delay-Differential Equations, *J. Differential Equations*, **72** (1988), 270-315.

[M-P,S] J. Mallet-Paret and H. Smith, The Poincaré-Bendixon Theorem for Monotone Feedback Systems, *J. Dynam. Differential Equations*, **2**(1990), 367-421.

[R] C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos.*, CRC Press Inc. (1994).