Global dynamics of neural nets with infinite gain.

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Abstract

We consider a model of neural and gene networks where the nonlinearities in the system of differential equations are discontinuous and piecewise constant. We develop a framework for study of such systems. As a first step we associate to the system a graph G on a hypercube and show how the collection of strongly connected components of G relates to the dynamics of the flow on the set of rays through the origin. In the second step we discuss the relationship between the invariant sets of the ray flow and the invariant sets of the original flow. We provide a sufficient condition for a one-to-one correspondence between these sets.

Finally, we study the class of binary networks within this framework. Under certain conditions we can determine the structure of an invariant set corresponding to the lowest strongly connected component of the hypercube graph.

Key words. Neural and gene networks, strongly connected components of an oriented graph, global dynamics.

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1 Introduction

The additive neural network

$$\dot{x}_i = -x_i + \sum_{j \neq i} w_{ij} f_j(x_j), \quad i = 1, \dots, n, \tag{1}$$

is a prototype of a system which, although composed of relatively simple units, may exhibit wide range of dynamical behavior. This model may not be a particularly faithful representation of the processes in the brain but is a good testing ground for methods which aim to describe dynamics of large systems of differential equations. The study of this model goes back to Grossberg [8, 9] and Hopfield [10]. The numbers w_{ij} were interpreted as synaptic connection weights and functions $f_j(x_j)$ are nonlinear sigmoidal gating functions with $f_j(0) = 0$. The *gain* of f_j is the derivative $f_j'(0)$. All of the results about the dynamics of these networks assume certain structure of the matrix $W = [w_{ij}]$ of connection weights, see Hopfield [10] for symmetric W, Atyia and Baldi [2] for circular matrix W and Gedeon and Fiedler [4] for the generalization of the symmetry assumption on W. To our knowledge there is no theory which would describe the dynamics in the general case. In this paper we propose a framework in which the dynamics for arbitrary matrix W can be studied in the limit of *infinite gain*. In this limit, the sigmoidal gating functions f_j become step functions

$$f_j(x_j) = \begin{cases} u_j & \text{for } x_j > 0 \\ -v_j & \text{for } x_j < 0 \end{cases} . \tag{2}$$

The simplification brought by this choice of nonlinearities comes from the fact that the right hand side of the equations (1) has a fixed value on the interior of any given orthant in $\mathbf{R}^n$ and the equations can be solved there explicitly. Under appropriate assumptions, solutions can be glued together at the boundary between the two orthants to form a C^0 flow. The difficulties come from the fact that the solution may traverse these boundaries infinitely many times and thus the long term prediction of the behavior remains difficult.

We wish to simplify the situation further. Under assumption **(H1)** below, we can associate to system (1) with nonlinearities (2) a *graph of the dynamics* G: Every orthant in $\mathbf{R}^n$ will be represented by a vertex in the graph and a boundary between two orthants by an edge. Every edge is oriented along the flow through the corresponding boundary between orthants. Observe, that there are many systems (3) which lead to the same graph G. Changing values v_j and u_j associated to the function f_j in (2) changes the graph only if these numbers change sign. Therefore the information that can be inferred from the graph about the dynamics of (3) must be valid for every system which gives rise to the same graph G. Our methods are applicable to a slightly more general set of equations, introduced by L. Glass [6]

$$\dot{x}_i = -\gamma_i x_i + \Lambda_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n. \tag{3}$$

The functions Λ_i depend only on the signs of the variables $x_1, \ldots, x_n$ and hence are constant on the interior of every orthant in $\mathbf{R}^n$.

In the chain of simplifications

$$\text{additive network (1)} \rightarrow \text{ infinite gain network (3)} \rightarrow \text{ graph } G$$

we are interested how we can use the information gained on the simpler level to understand the higher level. The first simplification is not as drastic as the second and can be handled by classical perturbation results. In this paper we study second part of this chain.

Some questions related to the correspondence between the graph G and dynamics of (3) were addressed by Glass and collaborators in [6] and [7]. The correspondence between stable equilibria of (1) for symmetric matrix of weights $W = [w_{ij}]$ and the stable fixed points of an asynchronous content addressable memory as the gain approaches infinity was studied in Hopfield [10]. His argument was corrected and completed by T. Troyer [17].

We also note that (3) is a generalization of $N-K$ *systems*, studied by S. Kauffman [11] as a model of complex interactions between genes and controlling proteins in a cell. This admittedly simplistic model assumes that every gene or protein is either active or inactive and so it can be modeled by a binary variable. There are N genes or proteins in the system. The phase space of the systems consists of 2^N vectors in $\mathbf{R}^N$, which are arranged as vertices of a hypercube in $\mathbf{R}^N$. The dynamics is simple; every unit changes its state based on the states of K other units, i.e.

$$x_i(t+1) = \Lambda_i(x_{i_1}(t), \ldots, x_{i_K}(t)). \tag{4}$$

Here Λ_i is a logical function of the same type as in (3).

To see the correspondence between (3) and (4) we integrate equations (3) in a fixed orthant $\mathcal{O}$ in $\mathbf{R}^n$. Since in $\mathcal{O}$ the values of Λ_i are constant, say $p_i := \Lambda_i(\mathcal{O})$, we get

$$x_i(t) = p_i + (x_i(0) - p_i)e^{-\gamma_i t}. \tag{5}$$

The point $p(\mathcal{O}) := (p_1, \ldots, p_n)$ is called the *target point* of the orthant $\mathcal{O}$. The dynamics in every orthant is linear and focused toward the target point.

We see now that (3) is an asynchronous version of the $N-K$ system; while solutions of (4) jump to their target points instantaneously, solutions of (3) approach the target point gradually.

2 Main results

In this paper we concentrate on two problems. First, we show how the properties of the graph G influence dynamics of (3) and, at the same time, we provide a framework within which questions about the dynamics of (3) can be successfully tackled.

In the second part we use this framework to investigate a particular class of systems (3) where all target points p have coordinates $\pm a$ for some a. We call these systems *binary systems*. This agrees with the choice of binary values S. Kauffman uses for $N-K$ systems. For this subclass we use the developed framework to characterize invariant sets of (3) for a class of graphs G.

2.1 Infinite gain networks.

We assume that

the value of $\Lambda_i(x_1, \ldots, x_n)$ does not depend on the sign of x_i (**H1**).

This assumption is satisfied for additive neural networks (1). To avoid unnecessary technical problems we also assume that

every target point p lies in the interior of an orthant (**H2**).

The assumption (**H1**) guarantees that the trajectories of (3) can be continued from orthant $\mathcal{O}_A$ to the next orthant $\mathcal{O}_B$, if the trajectory hits a codimension 1 hyperplane which separates $\mathcal{O}_A$ and $\mathcal{O}_B$. In this case we define the vector field on the wall to be the closure of the vector field in $\mathcal{O}_B$. We denote this C^0 piecewise smooth flow by $\Phi(x,t)$.

If the trajectory hits a codimension p hyperplane, $p \geq 2$,

$$H := \{x \in \mathbf{R}^n \mid x_{i_1} = \ldots = x_{i_p} = 0\},$$

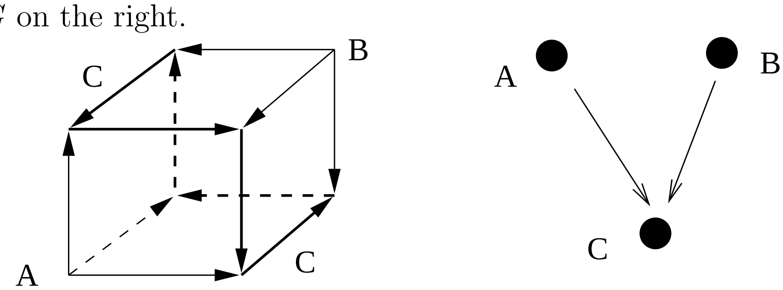
then the flow is not well defined, since this hyperplane is in the closure of more then 2 orthants. Our goal is to use the system (3) as an approximation of the system of the type (1). If S is an invariant for (3), we want to assert an existence of a similar invariant set $\tilde{S}$ for (1), for sufficiently large values of gains. For this reason we are looking for sufficiently stable invariant sets for (3), which we will be able to perturb back to (1). It is therefore reasonable to restrict our attention to the domain

$$D := \{x \in \mathbf{R}^n \mid \Phi(x,t) \notin H \quad \text{for all } t \text{ and } H\}.$$

2.2 Graph of the dynamics.

A *strongly connected component (SCC)* of an oriented graph G is a maximal set of vertices such that for every pair of vertices $u, v \in G$ there is path in the graph from u to v and the path from v to u. If G is connected, the strongly connected components form a partially ordered set. We illustrate this in an example below.

Figure 1: An oriented graph G on the left and the induced order on the set of connected components of G on the right.



We observe that every invariant set of (3) has to lie in the set of orthants corresponding to one SCC of the graph G. Furthermore, the order on SCC's induces order on the invariant sets; if G_0 is a strongly connected component above G_1 in the partial order then there is no connecting orbit from the invariant set corresponding to G_1 to invariant set corresponding to G_0. The order also suggests that the attractors with large basins of attraction are most likely lying in the orthants corresponding to minimal SCC in the partial order. Thus the first step in analyzing dynamics of a given system (3) would be to analyze these minima of the partial order, since they likely contain the attractors for (3). Furthermore, one would hope that the structure of the subgraph corresponding to a particular SCC has a strong influence on the type of the invariant set in the corresponding set of orthants.

We now examine this idea in the light of available results. We say that $S \subset G$ is an *attracting* set of vertices, if it is a strongly connected component of G and a minimum in the partial order. The corresponding set of orthants, $\mathcal{O}_S$, is an *attracting set of orthants*.

The simplest attracting set of vertices is a single vertex. If a vertex p of G is attracting, then the target point of the corresponding orthant $\mathcal{O}_p$ lies in $\mathcal{O}_p$. This target point is attracting all points in $\mathcal{O}_p$. Hence attracting vertex in the graph G implies existence of attracting fixed point in the dynamics of (3) and the correspondence between the graph and the dynamics is complete.

Next Theorem, due to Glass and Pasternack [7], examines this issue for cyclic attractor. An *N-dimensional cyclic attractor* C in G is an attracting cycle in G, which is not contained in any lower dimensional sub-cube of G.

Theorem 2.1 ([7]) *Given an N-dimensional cyclic attractor C in G, the corresponding set $\mathcal{O}_C$ either admits a unique stable periodic orbit, or all solutions in $\mathcal{O}_C$ converge to the origin.*

This result shows that the correspondence between the dynamics and structure of the subgraph of SCC is still good, but there are two possible outcomes. This means some systems (3) with cyclic attractor in G admit periodic orbit while others with the same graph G have no periodic orbit. On the closer inspection one realizes that this dichotomy is inevitable. For a given strongly connected component $S \subset G$, where S contains more then one vertex, there is always a selection of target points such that all trajectories in $\mathcal{O}_S$ converge to zero.

Does this mean that we have to give up hope for correspondence between the structure of SCC and the dynamics in the corresponding orthants? No, but we must address the special role of zero in the dynamics of (3).

Mestl et. al. [13] observed that all solutions of (3) starting on the same ray through the origin converge to a single point as $t \to \infty$. This allows us to separate radial dynamics from the dynamics of rays: to determine the long term behavior of solution through x we first determine the behavior of the ray on which x lies and then determine, whether the points along this ray converge to zero or not. We call the flow on the set of rays, induced by the flow of system (3), a *ray flow* (R-flow). Precise definition is presented in a later section.

The R-flow can be thought of as a flow on the $n-1$ dimensional unit sphere. Every orthant $\mathcal{O}$ intersects the unit sphere in a region $R_{\mathcal{O}}$. The graph G not only describes the direction of the flow of (3) from orthant to orthant, it also describes the direction of the R-flow from a region to region on the unit sphere. Since R-flow does not contain zero, it is easy to see, that Theorem 2.1 gives rise to the following result: if G has an N-dimensional cyclic attractor then the R-flow of (3) admits unique periodic orbit. There is a complete correspondence between the structure of the graph G and the dynamics of the R-flow. Since at the end we are not interested in dynamics of the R-flow, but in dynamics of the original problem (3) we have to address the correspondence between the invariant sets in these two flows.

To summarize, we propose the following approach to study dynamics of (3):

1. Determine SCC decomposition of G.

2. Deduce information about the dynamics of R-flow based on information about the structure of SCC's of the graph G.

3. By examining radial dynamics, determine how much of this information is valid for the flow of (3).

Our first Theorem below expresses the fact that SCC decomposition of the graph induces a corresponding decomposition of the invariant set of the R-flow. This result is not true for the original flow (3). Such a decomposition is called *Morse decomposition* and the invariant set, which lies in the orthants corresponding to a particular SCC, is called a *Morse set*.

Definition 2.2 The *Morse decomposition of a compact invariant set* S is a finite collection

$$\mathcal{M} = \{M(q) \mid q \in (\mathcal{P}, >)\}$$

of disjoint compact invariant sets, indexed by a strictly partially ordered finite set $\mathcal{P}$. For every $x \in S \setminus \bigcup_{q \in \mathcal{P}} M(q)$ there are $q, r \in \mathcal{P}$ with $q > r$ such that

$$\omega(x) \in M(r) \qquad \text{and} \qquad \alpha(x) \in M(q).$$

The sets $M(p)$ are called *Morse sets.*

Here $\omega(x)$ and $\alpha(x)$ denote omega and alpha limit sets of x respectively (see ([16])). The idea of a Morse decomposition was used for attractor of scalar delay equation by Mallet-Paret [12] and for cyclic feedback systems by Gedeon and Mischaikow [5].

Theorem 2.3 *Consider system (3) and assume (H1) and (H2). Then the ray flow admits a Morse decomposition*

$$\mathcal{PM} := \{PM(q) \mid q \in (\mathcal{P}, >)\}.$$

This Morse decomposition is determined by the SCC decomposition of the graph G.

The connection between the graph G and invariant sets of R-flow is close. There are topological methods which can benefit from this result and may be used to determine the character of the invariant set in the corresponding set of orthants based solely on the graph G. Such arguments may include fixed point theory, Wazewski retract theorem (see [3] II.2.3 or [18]) or Conley index theory (see Mischaikow [1] for review of the theory and Mischaikow et.al. [14] for results about existence of periodic orbits).

The third step of our strategy is to look at the radial dynamics. As we saw in Theorem 2.1 while all R-flows with cyclic attractor in the graph G admit a periodic orbit, some of the systems (3) have a periodic orbit and some do not. In the former case solutions of (3) along the periodic set of rays converge to a non-zero orbit while in the second case they converge to zero. Given a vertex P in the graph G we denote by $w_{in}(P)$ the set of incoming edges

$$w_{in}(P) = \{i \mid x_i = 0 \text{ along an edge pointed toward } P\}.$$

The set of outgoing will be denoted by $w_{out}(P)$. A vertex P is a *splitting vertex* if there are at least two outgoing edges from P. In Theorem 3.5 we find sufficient conditions for the first case to happen. As a corollary we get following generalization of the result of Glass and Pasternack ([7]).

Corollary 2.4 *Consider an attracting set $\mathcal{O}_A$ of orthants without a splitting vertex. Then*

1. *either there is a periodic orbit of (3) in $\mathcal{O}_A$*

2. *all solutions in $\mathcal{O}_A$ converge to zero.*

Furthermore, if for every orthant $\mathcal{O}$ in $\mathcal{O}_A$ the target point $p(\mathcal{O}) = (p_1, \ldots, p_n)$ satisfies

$$p_j^2 < \sum_{k \in w_{in}(\mathcal{O})} p_k^2$$

where $\{j\} = w_{out}(\mathcal{O})$, then the first part holds.

In the last part of the paper we apply our approach to binary dynamics.

Definition 2.5 We say that the system (3) is a *binary system* if the set of target points is a subset of a hypercube with vertices $(\pm a, \pm a, \ldots, \pm a)$ for some a.

It is easy to see that one can rescale the phase space so that $a = 1$.

We study the R-flow associated to binary system (3). In Theorem 4.1 we describe explicitly how this flow acts on the incoming walls of an orthant $\mathcal{O}$. In Proposition 4.2 we partially describe how the local descriptions in individual orthants can be pieced together. We then define a class of *simple SCC*. We show that every invariant set lying in orthants corresponding to a simple SCC, consists only of periodic orbits whose structure can be described explicitly. No chaotic dynamics is possible in the invariant set corresponding to a simple SCC.

Theorem 2.6 *An invariant set of (3) corresponding to the simple SCC in G is a finite (or empty) collection of periodic orbits.*

A set of orthants $\mathcal{O}_A$ is a *one-split attractor* if it is attracting set of orthants and there is a unique splitting vertex in the subgraph $A \subset G$.

Corollary 2.7 *For a binary system the invariant set in a one-split attractor is a finite (or empty) set of periodic orbits.*

3 Dynamics of infinite gain networks and the graph G

3.1 The R-flow

As we saw in the introduction, the trajectories of (3) are straight lines inside every orthant and may have corners on the boundaries between the orthants. We call such a boundary a *wall*. Each wall corresponds to an edge of the graph G. Each wall is either *incoming* or *outgoing* relative to the orthant $\mathcal{O}_A$ depending on whether the corresponding edge points toward or away from A.

We may compute a transition function from an incoming wall W_1 to an outgoing wall W_2 through an orthant $\mathcal{O}$. Obviously, W_1 and W_2 must be parts of the boundary of $\mathcal{O}$. This function takes an initial value on an W_1 and associates to it the intersection of the solution with W_2. Assume that on the wall W_2 we have $x_j = 0$ and that the target point of an orthant $\mathcal{O}$ is $p = (p_1, \ldots, p_n)$. Solving the j-th equation for the time of transition t^* and then substituting to the other equations we find that

$$y_i = \frac{x_i(0) - (p_i/p_j)x_j(0)}{1 - x_j(0)/p_j}. \tag{6}$$

In the vector notation (see [7])

$$\mathbf{y} = M_{\mathcal{O}}(\mathbf{x}) = \frac{C\mathbf{x}(0)}{1 + \mathbf{c}^t\mathbf{x}(0)}, \tag{7}$$

where $C \in \mathbf{R}^{n\times n}$ and the transposed vector $\mathbf{c}$ has zero entries except $\mathbf{c}_j = -1/p_j$. The assumption (**H2**) implies that $\mathbf{c}^t\mathbf{x}(0) > 0$. Observe that the formula (6) does not depend on the wall W_1 in other way than initial condition. So this formula is valid for the transition from any incoming wall to the outgoing wall W_2.

The map $M_{\mathcal{O}}$ is a linear fractional transformation and composition for two such transformations is again a linear fractional transformation.

Consider two points, $x_1 = k_1 v$ and $x_2 = k_2 v$, on a ray starting at the origin. The ray is the set of all positive multiples of a given vector, $\{x \mid x = kv, k > 0\}$. Assume that the ray lies on a wall W_1. Then it is easy to see that $y_1 := M_{\mathcal{O}}(x_1)$ and $y_2 := M_{\mathcal{O}}(x_2)$ are again collinear and lie on a ray Cv.

Furthermore the linearity of the flow in $\mathcal{O}$ guarantees that the trajectories starting at x_1 and x_2 are collinear. By this we mean the following: given a point $z = \Phi(x_1, t)$, $0 \le t \le t^*(x_1)$ where $t^*(x_1)$ is given by $y_1 = \Phi(x_1, t^*(x_1))$, there is a point $w = \Phi(x_2, \bar{t})$, $0 \le \bar{t} \le t^*(x_2)$, such that z and w are collinear. Notice, that $t^*(x_1) \ne t^*(x_2)$ in general. Since the flow $\Phi(x,t)$ is linear in $\mathcal{O}$, the flow lines of $\Phi(x,t)$ starting at the ray through x span the plane containing rays through x and Cx. We rescale time in flow $\Phi(x,t)$ so that the points initially on a ray stay on a ray throughout the interior of the orthant $\mathcal{O}$. The rescaling can be made continuous by the continuous dependence on initial conditions. After rescaling we project the flow $\Phi(x,t)$ onto the unit sphere to obtain the *ray flow (R-flow)* $\varphi(x,t)$ on a subset $\mathcal{D}$ of S^{n-1}. The set $\mathcal{D}$ is defined as

$$\mathcal{D} := \{z \in S^{n-1} \mid [\varphi(z,t)]_i = 0 \text{ for at most one } i \in \{1, \dots, n\}\},$$

where $[\varphi(z,t)]_i$ is the i-th component of the vector $\varphi(z,t)$. The walls in $\mathbf{R}^n$ divide the sphere S^{n-1} into regions, each of which lies in one orthant. Every $z \in \mathcal{D}$ corresponds to a ray in $\mathbf{R}^n$. Most of the time we will work with the maps mapping the rays in incoming walls to the rays on the outgoing walls, rather then directly with R-flow. We again emphasize that these maps are linear

$$y = Cx. \tag{8}$$

Let $\mathcal{G} = \{G(q) \mid q \in (\mathcal{P}, >)\}$ be a decomposition of the graph G into strongly connected components $G(q)$, ordered by a partial order $(\mathcal{P}, >)$.

Definition 3.1 Given the graph decomposition $\mathcal{G} = \{G(q) \mid q \in (\mathcal{P}, >)\}$,W we denote by $\mathcal{O}(i)$ the set of orthants corresponding to the subgraph $G(i)$. We set

$$\mathcal{PM} := \{PM(q) \mid q \in (\mathcal{P}, >)\}$$

where $PM(i)$ is the maximal invariant set in $\mathcal{D} \cap Int\, \mathcal{O}(i)$.

Proof of Theorem 2.3

Let $x \in S \setminus \bigcup_{i\in\mathcal{P}} PM(i)$. This implies that $\omega(x) \in PM(j)$ and $\alpha(x) \in PM(i)$ for $j \ne i$. The set of orthants $\mathcal{O}(i) \cup \bigcup_{k<i} \mathcal{O}(k)$ is positively invariant under the R-flow φ by the construction of $\mathcal{G}$. This set must contain $\mathcal{O}(j)$ since solution $x(t)$ connects $PM(i)$ to $PM(j)$. By the construction of the partial order $(\mathcal{P}, >)$ this implies $i > j$ in $\mathcal{P}$. $\square$

3.2 Convergence to zero

We first recall a result of Mestl *et. al.* [13].

Lemma 3.2 (Theorem 3,[13]) *The distance between points $\Phi(x_1, t)$ and $\Phi(x_2, t)$, where $x_2 = kx_1$, $x_1, x_2 \in D$ goes to zero as the time goes to infinity.*

Proof. Assume that the trajectories $\Phi(x_1, t)$ and $\Phi(x_2, t)$ traverse through a sequence of orthants $T = \{\mathcal{O}_1, \dots, \mathcal{O}_n\}$. Then the corresponding mapping given by the composition of individual transition maps (7) is a fractional linear map $M_T = \frac{S_n \mathbf{x}}{1+\mathbf{s}^t\mathbf{x}}$, where $S_n = C_n \dots C_1$, and $\mathbf{s}^t\mathbf{x} = \mathbf{c}_1^t\mathbf{x} + \mathbf{c}_2^t C_1 \mathbf{x} + \dots + \mathbf{c}_n^t S_{n-1}\mathbf{x}$. Note that $\mathbf{s}^t\mathbf{x} \to \infty$ as $n \to \infty$. Therefore, as $n \to \infty$

$$\lim_{n\to\infty} \frac{\mathbf{x}_1^{(n)}}{\mathbf{x}_2^{(n)}} = \lim_{\mathbf{s}^t\mathbf{x}\to\infty} \frac{1 + k\mathbf{s}^t\mathbf{x}}{k(1+\mathbf{s}^t\mathbf{x})} = 1.$$

By the continuity of the flow in $\mathcal{D}$ this implies the result. □

Corollary 3.3 *Fix a ray $y = kx, k > 0$. Then all solutions of (3) starting at initial values on this ray either converge to an unique nonzero solution, or converge to zero.*

Proof. In view of Lemma 3.2 all we need to show is the boundedness of all solutions. This follows from the fact that the functions $|\Lambda_i(x)|$ are uniformly bounded on $\mathbf{R}^n$. □

It is clear, that the invariant set $\mathcal{S}$ for the R-flow corresponds to an invariant set S for the flow Φ if the solutions on the set of rays, corresponding to $\mathcal{S}$, do not converge to zero.

We present now a sufficient conditions for this to happen. Consider a Morse set $PM(q)$ and the corresponding subgraph $G(q)$. Fix a vertex $A \in G(q)$ and assume that $i \in w_{in}(A)$ and $j \in w_{out}(A)$.

Definition 3.4 We say that the transition $i \to j$ at a vertex A is an *expanding transition* if for every $x \in PM(q) \cap \{x_i = 0\}$.

$$\sum_{k\neq i,j} p_k x_k > 0 \tag{9}$$

$$p_j^2 < \sum_{k\neq j} p_k^2 \tag{10}$$

where $p = (p_1, \dots, p_n)$ is the target point in orthant $\mathcal{O}_A$.

We want to remark that

- the first condition is automatically satisfied if every target point is in the neighboring orthant;
- the second condition is easy to check and does not require knowledge about the invariant set $PM(q)$.

Theorem 3.5 *If all transitions in $G(q)$ are expanding then the Morse set $PM(q)$ is homeomorphic to an invariant set $M(q)$ of the system (3).*

Proof. We shall show that if the transition $i \to j$ at vertex A is expanding, then

$$||M_A(x) \geq ||x|| \tag{11}$$

for $||x||$ sufficiently small (see (7)). This implies that trajectories in $M(q)$ cannot converge to the origin. By Corollary 3.3 every trajectory in $PM(q)$ has a homeomorphic image in $M(q)$.

It remains to show (11). By (6) $M_A(\mathbf{x}) = \frac{C\mathbf{x}}{1+x_j/(-p_j)}$. Also recall that $x_j/(-p_j) = |x_j|/|p_j| > 0$ since the transition is through the wall $x_j = 0$. Then

$$\begin{aligned} ||C\mathbf{x}|| &= (x_1 - (p_1/p_j)x_j)^2 + \ldots + ((p_i/p_j)x_j)^2 + \ldots + (x_n - (p_n/p_j)x_j)^2 \\ &= (\sum_{k \neq j} (p_k/p_j)^2) x_j^2 + \sum_{k \neq i,j} x_k^2 + 2\frac{|x_j|}{|p_j|}(\sum_{k \neq i,j} p_k x_k) \end{aligned}$$

Let us denote $\epsilon := |\frac{x_j}{p_j}|$ and assume that $\epsilon << 1$. This corresponds to $\mathbf{x}$ close to zero. We compute

$$\begin{aligned} ||M_A(\mathbf{x})|| &= \frac{1}{1+\epsilon}[(\sum_{k \neq j} p_k^2)\epsilon^2 + \sum_{k \neq i,j} x_k^2 + 2\epsilon(\sum_{k \neq i,j} p_k x_k)] \\ &= (1 - \epsilon + \epsilon^2)[(\sum_{k \neq j} p_k^2)\epsilon^2 + \sum_{k \neq i,j} x_k^2 + 2\epsilon(\sum_{k \neq i,j} p_k x_k)] \\ &= \sum_{k \neq i,j} x_k^2 + \epsilon[2(\sum_{k \neq i,j} p_k x_k) - \sum_{k \neq i,j} x_k^2] \\ &+ \epsilon^2[\sum_{k \neq j} p_k^2 + \sum_{k \neq i,j} x_k^2 - 2(\sum_{k \neq i,j} p_k x_k)]. \end{aligned}$$

Let us denote $L := 2(\sum_{k \neq i,j} p_k x_k) - \sum_{k \neq i,j} x_k^2$ and observe that

$$||\mathbf{x}|| = \sum_{k \neq i,j} x_k^2 + \epsilon^2 p_j^2.$$

Then we have

$$\begin{aligned} ||M_A(\mathbf{x})|| &= \sum_{k \neq i,j} x_k^2 + L(\epsilon - \epsilon^2) + \epsilon^2 \sum_{k \neq j} (p_k)^2 + O(\epsilon^3) \\ &> \sum_{k \neq i,j} x_k^2 + \epsilon^2 \sum_{k \neq j} (p_k)^2 \\ &> \sum_{k \neq i,j} x_k^2 + \epsilon^2 p_j^2 \\ &= ||\mathbf{x}|| \end{aligned}$$

where we used that for sufficiently small $\mathbf{x}$ the quantity $L > 0$ if and only if (9) holds. The second inequality follows from (10). □

Proof of Corollary 2.4 Let us fix a wall W of an orthant $\mathcal{O}$ in the set $\mathcal{O}_A$, which corresponds to an edge in the subgraph $G(A)$. We consider an R-flow acting on the intersection

$U := W \cap \mathcal{O} \cap \mathcal{D}$. Since $\mathcal{O}_A$ is an attracting set of orthants, it is easy to see that W is a Poincaré section of the Morse set $M := PM(A)$ under the R-flow. By (8) the map from an incoming wall to an outgoing wall in R-flow is $\mathbf{y} = C\mathbf{x}$. A Poincaré map π on U is a finite composition of such maps and thus it is linear. Since $\mathcal{O}_A$ is an attracting set of orthants, π maps the boundary of U into U. By the Brouwer fixed point theorem ([15]) there is a fixed point of π in U. Since π is linear there is at most a finite number of fixed points in U, which correspond to eigenvectors of π. These correspond to periodic orbits of the R-flow.

By Corollary 3.3 for every such periodic orbit of the R-flow there is either a corresponding periodic orbit of (3) or solutions starting on the corresponding rays go to zero. Since there is no splitting vertex in $G(A)$ the target point of every orthant $\mathcal{O}$ in $\mathcal{O}_A$ lies in an orthant $\mathcal{V} \subset \mathcal{O}_A$, which differs from $\mathcal{O}$ in one variable. Hence the condition (9) is satisfied for all $x \in PM(\mathcal{O}_A)$. The result now follows. $\square$

4 Binary dynamics

In this section we study binary systems. We denote by X_i the hyperplane given by $\{x_i = 0\}$. Let $\mathcal{O}$ be an arbitrary orthant. We will represent the intersection $X_i \cap \mathcal{O}$ and the sphere S^{n-1} as an $n-2$ dimensional simplex. In $\mathbf{R}^3$ this amounts to representation of S^2 as an octagon with vertices in vectors $(\pm 1, 0, 0), (0, \pm 1, 0)$ and $(0, 0, \pm 1)$. The intersections with all X_i are line segments i.e. 1-simplices.

Each $n-2$ dimensional simplex $X_i \cap \mathcal{O}$ in $\mathbf{R}^n$ can be parameterized as

$$\{x \in \mathcal{O} \mid x_i = 0, \sum_{j \neq i} |x_j| = 1\}.$$

We will use this parameterization in what follows.

To describe the dynamics of the R-flow in an orthant $\mathcal{O}$ we need to describe a flow induced map from an arbitrary incoming simplex to an arbitrary outgoing simplex of $\mathcal{O}$.

Proposition 4.1 *Fix an arbitrary orthant $\mathcal{O}_P \subset \mathbf{R}^n$ and denote the corresponding vertex in the graph G by P. Choose $i \in w_{in}(P)$ and $j \in w_{out}(P)$. Then the simplex*

$$I := \{|x_j| \leq |x_k| \mid k \in (w_{out} \setminus \{j\})\},$$

$I \subset X_i \cap \mathcal{O}_P$, is mapped into the simplex

$$O := \{|x_i| \leq |x_k| \mid k \in (w_{in} \setminus \{i\})\}$$

$O \subset X_j \cap \mathcal{O}_P$ by the R-flow. If $(w_{out} \setminus \{j\}) = \emptyset$ then $I = X_i \cap \mathcal{O}_P$ and if $(w_{in} \setminus \{i\}) = \emptyset$ then $O = X_j \cap \mathcal{O}_P$.

Proof. To every $u \in X_i \cap \mathcal{O}_P$ and $k \in w_{out}$ we can associate the time

$$t_k(u) := min\{t \mid \varphi(u, t) \in X_k\}.$$

Observe that any $u \in X_i \cap \mathcal{O}_P$ such that $\varphi(x,t)$ exits the orthant $\mathcal{O}_P$ through the face $X_j \cap \mathcal{O}_P$ must satisfy $t_j(u) \leq t_k(u)$ for all $k \in (w_{out} \setminus \{j\})$. We compute the times $t_k(u)$ and $t_j(u)$ using (5). By setting $x_j = 0$ and $x_k = 0$, respectively, we get

$$e^{-t_j} = -\frac{p_j}{x_j(0) - p_j}, \quad e^{-t_k} = -\frac{p_k}{x_k(0) - p_k}$$

where $p_j, p_k = \pm 1$ are components of the target point in the orthant $\mathcal{O}_P$. Since both j and k are in $w_{out}(P)$ we have that $x_j(0)p_j < 0$ and $x_k(0)p_k < 0$. Then

$$e^{-t_l} = \frac{|p_l|}{|x_l(0)| + |p_l|} = \frac{1}{|x_l(0)| + 1}$$

for both $l = j$ and $l = k$. Thus $e^{t_j} = |x_j(0)| + 1$ and $e^{t_k} = |x_k(0)| + 1$. Therefore the solutions starting at simplex $I = \{|x_j| \leq |x_k| \mid k \in (w_{out} \setminus \{j\})\}$ exit the orthant $\mathcal{O}_P$ through the wall X_j.

Now we look at $X_j \cap \mathcal{O}_P$ and see where the points landing on $X_j \cap \mathcal{O}_P$ came from. By (6)

$$x_k = \frac{x_k(0) - p_k \frac{x_j(0)}{p_j}}{1 - \frac{x_j(0)}{p_j}}.$$

We observe that $x_j(0)/p_j = -|x_j(0)|$ since $j \in w_{out}$ and $p_j = \pm 1$. Thus

$$x_k = \frac{x_k(0) + p_k|x_j(0)|}{1 + |x_j(0)|}. \tag{12}$$

For every $k \in w_{in}$ we have that $x_k(0)p_k > 0$ and for such k we can write

$$|x_k| = \frac{|x_k(0)| + |x_j(0)|}{1 + |x_j(0)|}. \tag{13}$$

If the point x came from the incoming hyperplane $X_i \cap \mathcal{O}_P$ then

$$|x_i| = \frac{|x_j(0)|}{1 + |x_j(0)|}$$

and from (13)

$$|x_k| = \frac{|x_k(0)|}{1 + |x_j(0)|} + |x_i| \geq |x_i|.$$

Therefore the set in $X_j \cap \mathcal{O}_P$ which got mapped onto from $X_i \cap \mathcal{O}_P$ is the simplex O. This proves the Proposition. $\square$

The Proposition 4.1 describes how the simplices are mapped from an incoming hyperplane of an orthant to an outgoing hyperplane of an orthant. In order to understand the invariant sets in a collection of orthants we need to compose the maps in successive orthants. While this process is in general very complicated and is the essence of complicated dynamics in systems (3), we will get a partial answer. Every simplex S which we consider lies in the intersection of some X_i and our representation of the unit sphere S^{n-1}. As such, it is defined by a set of inequalities between n quantities $|x_1|, |x_2|, \ldots, |x_n|$. We associate to every simplex S its defining set of inequalities $\mathcal{I}(S)$. Since $S \subset X_i$, for some i, the set $\mathcal{I}(S)$ should include inequalities $|x_i| \leq |x_k|$ for all k, since $|x_i| = 0$ on X_i. Since we consider these inequalities trivial we exclude them from $\mathcal{I}(S)$.

Proposition 4.2 *Consider a simplex $S \subset (\mathcal{O}_P \cap X_i)$ given by a set of inequalities $\mathcal{I}(S)$. Then the image of S by the R-flow in the orthant $\mathcal{O}_P$ is a subset of a simplex $Q \subset X_j$ whose set $\mathcal{I}(Q)$ is constructed from $\mathcal{I}(S)$ in the following way:*

- *drop from $\mathcal{I}(S)$ all inequalities involving x_j*
- *drop from $\mathcal{I}(S)$ all inequalities $|x_l| \leq |x_m|$ where $l \in w_{in}(P), m \in w_{out}(P)$*
- *add inequalities $|x_i| \leq |x_k|$ for all $k \in w_{in}(P) \setminus \{i\}$*

Proof. All inequalities involving x_j are dropped since simplex Q lies in X_j.

The addition of the inequalities $|x_i| \leq |x_k|$ for all $k \in w_{in}(P) \setminus \{i\}$ follows from Proposition 4.1.

Recall that if $k \in w_{out}(P)$ then $x_k(0)p_k < 0$ and if $k \in w_{in}(P)$, then $x_k(0)p_k > 0$, where p is the target point of $\mathcal{O}_P$. Then from (12) we have

$$|x_k| = \frac{|x_k(0)| + |x_j(0)|}{1 + |x_j(0)|}$$

for $k \in w_{in}(P)$ and

$$|x_k| = \frac{|x_k(0)| - |x_j(0)|}{1 + |x_j(0)|}$$

for $k \in w_{out}(P)$. Therefore, if $|x_l(0)| \leq |x_m(0)|$ and $l \in w_{out}(P)$ we have $|x_l| \leq |x_m|$. Also, if $l \in w_{in}(P)$ and $m \in w_{in}(P)$, we have $|x_l| \leq |x_m|$. If however $|x_l(0)| \leq |x_m(0)|$ and $l \in w_{in}(P)$ and $m \in w_{out}(P)$, we cannot make any statement about $|x_l|$ and $|x_m|$ in general. By dropping all these inequalities we get a simplex Q which contains the image of S. $\square$

We shall define a class of strongly connected components of G for which we can determine the structure of the corresponding invariant set directly from the subgraph of SCC.

A closed cycle in the oriented graph G will be called a *loop*. The set $N(A)$ is the set of all edges adjacent to a vertex A. Given a subgraph $L \subset G$ we denote by $w_{in}(P)_{/L}$ the set of edges in the subgraph L which points to a vertex P. Similarly we define the set $w_{out}(P)_{/L}$ as the set of outgoing edges of P in the subgraph L.

Definition 4.3 We say that a loop L in the strongly connected component H is *simple* if, given any splitting vertex A and any other vertex $Q \in L$ we have

1. if $N(Q) \cap L \cap (w_{out}(A)_{/H}) = \emptyset$ then either $(w_{out}(A)_{/H}) \subset w_{in}(Q)$ or $(w_{out}(A)_{/H}) \subset w_{out}(Q)$.
2. if $\{j\} \in N(Q) \cap L \cap (w_{out}(A)_{/H})$ and the edge corresponding to $x_j = 0$ goes from vertex Q to vertex $\bar{Q}$, then either $(w_{out}(A)_{/H}) \subset w_{out}(Q)$ or $(w_{out}(A)_{/H}) \subset w_{in}(\bar{Q})$.

We say that the strongly connected component H is simple if every closed loop in H is simple.

Proposition 4.4 *Let A be a splitting vertex in a simple strongly connected component $H \subset G$. Assume that $x(t)$ is a solution of R-flow of a binary system (3), which leaves the orthant $\mathcal{O}_A$ through the hyperplane X_j at $t = 0$, for some j.*

If $x(t)$ reenters the orthant $\mathcal{O}_A$ again at $t = T$, it will subsequently leave $\mathcal{O}_A$ through X_s, where x_s is the last variable out of all variables in $w_{out}(A)_{/H}$ which has changed the sign along the solution $x(t)$, $0 \leq t \leq T$.

Proof. The solution $x(t)$ between time 0 and time T sweeps through a set of orthants, whose corresponding edges in the graph H form a loop L. Since the first edge of the loop L corresponds to crossing the hyperplane $x_j = 0$, there must be at least one other edge corresponding to crossing $x_j = 0$, before the loop L closes up. If there is an edge $x_l = 0$ in the loop L, with $l \neq j$, then it must occur even number of times in L. In any case, there is at least one edge in L, apart from the first edge $x_j = 0$, which corresponds to $x_l = 0$, for some $l \in w_{out}(A)_{/H}$. Assume that $x_s = 0$ is the last such edge in L and that the edge $x_s = 0$ goes from a vertex Q to a vertex $\bar{Q}$. Let the next edge after $x_s = 0$ along L correspond to $x_u = 0$. By Definition 4.3.2 either $(w_{out}(A)_{/H}) \subset w_{out}(Q)$ or $(w_{out}(A)_{/H}) \subset w_{in}(\bar{Q})$. In the first case by Proposition 4.1 the simplex which maps onto $X_s \cap \mathcal{O}_Q$ is a subset of

$$S := \{|x_s| < |x_j| \mid \text{ for all } j \in w_{out}(A)_{/H}\}. \tag{14}$$

In the second case, the Proposition 4.1 implies that the simplex which maps onto $X_u \cap \mathcal{O}_{\bar{Q}}$ is again a subset of the simplex S.

By assumption there is no other transition $x_k = 0$ with $k \in w_{out}(A)_{/H}$ in the remaining part of the loop L. Therefore, in every remaining vertex Q along the loop, the first case of Definition 4.3 applies. By Proposition 4.2, the inequalities (14) are preserved at every such vertex. Proposition 4.1 now implies that the solution $x(t)$ will exit orthant $\mathcal{O}_A$ through X_s upon arrival at $t = T$. The proof is complete.

□

Proof of Theorem 2.6 Fix a simple strongly connected component H of G. Let $x(t)$ be an arbitrary trajectory in the invariant set Inv$H$. Let $U(x)$ be a sequence of of orthants through which the solution $x(t)$ passes for $t \geq 0$ and let $\Upsilon(x)$ be the corresponding sequence of vertices in the graph H. If x travels through an orthant $\mathcal{O}_A$ corresponding to a splitting vertex A, in each subsequent visit, by Proposition 4.4, its trajectory is completely determined by the previous visit in orthant $\mathcal{O}_A$. Since there are finitely many vertices and edges in the subgraph H, the sequence $\Upsilon(x)$ must be periodic. Furthermore, its length is bounded above by product of number of vertices and number of edges.

For every periodic sequence of orthants visited we can use the argument of the proof of Corollary 2.4. The corresponding Poincare map is linear and there is at most finitely many periodic orbit of R-flow passing through the fixed periodic sequence of orthants. Since the number of admissible periodic sequences of orthants is bounded, as both their length is bounded and the number of ortants in H is bounded, there may be at most finitely many periodic orbits in Inv$H$ for the R-flow. By Corollary 3.3 the number of periodic orbits in the full flow of (3) is less or equal to the number of periodic orbits in R-flow. This finishes the proof. □

Proof of Corollary 2.7

The subgraph corresponding to a one-split attractor is a simple SCC , since the Definition 4.3.2 always applies with $w_{out}(A) \subset w_{in}(\bar{Q})$. $\square$

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