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Structure and dynamics of artificial neural networks

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1 Introduction

In this paper we consider an analog neural network

$$\dot{x}_i = -x_i + \sum_{j=1}^{n} w_{ij} f_j(x_j) \tag{1}$$

$i = 1, \dots, n$, where without loss of generality we assume $w_{ii} = 0$. Also, we consider a generalized Lotka-Volterra system

$$\dot{x}_i = x_i(c_i - \sum_{j=1}^{n} w_{ij} f_j(x_j)) \tag{2}$$

defined on the positive orthant $\mathbf{R}^{n+} = \{x \in \mathbf{R}^n \mid x_i \geq 0 \text{ for all } i\}$, where c_i, $i = 1, \dots, n$ are constants. In fact both of these systems belong to the class of equations

$$\dot{x}_i = a_i(\mathbf{x})(\gamma_i(x_i) - \sum_{j=1}^{n} w_{ij} f_j(x_j)) \tag{3}$$

where $w_{ii} = 0$. We assume that

$$\frac{\partial f_j}{\partial x_j} > 0 \text{ for all } j. \tag{H1}$$

and the functions $a_i(\mathbf{x})$ are nonnegative,

$$a_i(\mathbf{x}) \geq 0 \qquad \text{for all } \mathbf{x} \in \mathbf{R}^n \text{ and all } i. \tag{H2}$$

The main goal of this paper is to discuss correspondence between the structure of the matrix $W := [w_{ij}]$ and the dynamics of the system (3). We first describe motivation of the problem, then review the history and related results. We also want to point out related results in combinatorial matrix theory.

Systems of type (1) can be considered as a model neural network consisting of n neurons. Each x_i represents an activation level of the corresponding neuron. The stable equilibria of the system (1) represent stored memory. The trajectories, starting sufficiently close to an asymptotically stable equilibrium of (1), will converge to that equilibrium. From the point of view of a neural networks this models retrieval of a stored memory pattern upon presentation of a partial, corrupted pattern and preservation of this pattern in short term memory. This interpretation then leads to applications of systems (1) to parallel memory storage, content-addressable memory and global pattern formation. Models of short term memory and global pattern formation using systems of type (1) and (3) go back to work of Grossberg [1973,1978b]. These papers focus on proving general convergence theorems for certain classes of additive neural networks. Indeed, when trying to design a system which could serve as a memory storage, the fundamental question is whether the dynamics of the system is convergent or at least quasi-convergent (Hirsch [1989]). The system is *convergent*, if every trajectory of the system converges to an equilibrium. To show that the system is convergent, one must prove that equilibria are isolated. Since this is usually a nontrivial problem, we need more general notion of

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convergence. We say that the system is *quasi-convergent* , if every trajectory of the system converges to the set of equilibria. The system is *almost quasi-convergent* if trajectory of almost every initial value converges to the set of equilibria. Only after the question of convergence has been resolved one can actually try to design the system so that the equilibria are at the desired places, or, using again neural networks terminology, to design a system, which stores desired memory patterns.

Convergence theorems were proved for various generalizations of additive neural networks by Grossberg [1969a,1969b,1971] and for a system analogous to system (3), by Cohen and Grossberg [1983]. Our result below is a generalization of their result.

The *structure matrix* of the system (3) is matrix W. To every $n \times n$ matrix W we associate a digraph $D(W)$ containing n vertices and a directed edge e_{ij} from point i to point j if, and only if, $a_{ij} \neq 0$, $i \neq j$. Paths and cycles in $D(W)$ are defined in the usual way.

Definition 1.1. Matrix W is combinatorially symmetric, if $w_{ij} \neq 0$ implies $w_{ji} \neq 0$.

Definition 1.2. Matrix W is sign symmetric, if $w_{ij} \neq 0$ implies $w_{ij} w_{ji} > 0$.

Observe that if W is combinatorially symmetric, $D(W)$ gives rise to a undirected graph $G(W)$ by identifying the edges e_{ij} and e_{ji}.

Definition 1.3. The system (3) is *cooperative* if for any $i \neq j$, $w_{ij} \geq 0$ and *competitive* if for all $i \neq j$, $w_{ij} \leq 0$.

Properties of structure matrix W, which have nontrivial consequences for dynamics of (3) are symmetry, non-negativity and a simple structure of matrix W (i.e. tridiagonality). We shall illustrate this in several examples and then generalize some of these results.

Example 1.4 (Cohen and Grossberg (1983)). *Consider system (3) on* $\mathbf{R}^{n+}$. *Assume that the function* $a_i(\mathbf{x})$ *is a function of first variable only,* $a_i(\mathbf{x}) = a_i(x_i)$, *and* $a_i(x_i) > 0$ *for* $x_i > 0$. *Assume also (H1) and that functions* $f_j \geq 0$. *Furthermore, let* W *be symmetric and nonnegative.*

Then the system (3) admits a Lyapunov function and every bounded trajectory converges to the set of equilibria.

Example 1.5 (Hirsch (1982,1985)). *Consider system (1). Assume (H1), and that* W *is sign symmetric. Assume further that* W *has a* loop property*, which means that along every closed loop in graph* $G(W)$ *the product of matrix elements corresponding to the edges is positive.*

Then the system is almost quasi-convergent.

We want to remark that this result is a combination of result of Hirsch [1982,1985], which states that cooperative systems are almost quasi-convergent, and an abstract combinatorial result, which says that for every sign symmetric matrix W with the loop property, there is a change of variables which makes W a nonnegative matrix. (Eschenbach and Johnson [1990]). The formulation of loop property is due to Hirsch [1987] and Smith [1988].

Example 1.6 (Smilie (1984), Smith (1991)). *Consider system (1). Assume (H1), and that* W *is sign symmetric and tridiagonal, i.e.* $w_{ij} = 0$ *unless* $j = i + 1$ *or* $j = i - 1$. *Furthermore, assume that* W *is irreducible, i.e. the graph* $G(W)$ *has one component.*

Then (1) is convergent.

We mention briefly, that there are related results about *cyclic feedback systems.* These are systems of the form (3) where $w_{ij} = 0$ unless $j = i - 1$. Interested reader can find these results in Mallet-Paret and Smith [1990], Gedeon and Mischaikow [1995] and Gedeon [1998].

2 Main results

We present our results in two forms. We present dynamical result for system (3) and then the related abstract combinatorial result for matrix W. We first list additional assumptions.

There is a spanning tree T of $G(W)$ such that if $e_{ij} \in T$ then $w_{ij} w_{ji} > 0$. **(H3)**

Observe that this is slightly weaker assumption then sign symmetricity of W.

A *chord cycle* C_T, corresponding to a chord of tree T in the graph $G(W)$, is a collection of edges $e_{i_1 i_2}, e_{i_2 i_3}, \ldots, e_{i_k i_1}$ such that all edges, except one, are in T. Note, that every cycle C in $G(W)$ corresponds to two cycles $C^1 = \vec{e}_{i_1 i_2}, \vec{e}_{i_2 i_3}, \ldots, \vec{e}_{i_k i_1}$ and $C^2 = \overleftarrow{e}_{i_1 i_2}, \overleftarrow{e}_{i_2 i_3}, \ldots, \overleftarrow{e}_{i_k i_1}$ in the oriented graph $D(W)$, where the arrows indicate the orientation of the edges. To every cycle $C_T \in G(W)$ we associate two numbers:

$$p(C_T) := \Pi_{C_T^1} w_{ij} \qquad \text{and} \qquad \bar{p}(C_T) := \Pi_{C_T^2} w_{ji}.$$

We assume that

$$p(C_T) = \bar{p}(C_T) \qquad \text{for all chord cycles } C_T. \tag{H4}$$

Following is a generalization of the result in Fiedler and Gedeon [1997].

Theorem 2.1. *Consider system (3) with assumptions (H1-H4). Then there exists a Lyapunov function $V : \mathbf{R}^n \to \mathbf{R}$, non-increasing along the trajectories and strictly decreasing along all non-equilibrium trajectories of (3).*

Observe that this result replaces the symmetry hypothesis on the structure matrix W in Cohen and Grossberg [1983] (see also Hopfield [1982,1984]). Considerable effort was spent trying to remove or weaken this hypothesis. That this hypothesis cannot be removed completely is known for some time. Consider the May and Leonard [1975] model of the voting paradox

$$\begin{aligned} \dot{x}_1 &= x_1(1 - x_1 - ax_2 - bx_3) \\ \dot{x}_2 &= x_2(1 - bx_2 - x_2 - ax_3) \\ \dot{x}_3 &= x_3(1 - ax_1 - bx_2 - x_3). \end{aligned} \tag{4}$$

Grossberg [1978a] and Schuster *et. al.* [1979] considered the case $b > 1 > a$ and $a + b > 0$. They proved that all positive trajectories, except the uniform trajectories $x_1(0) = x_2(0) = x_3(0)$, persistently oscillate as $t \to \infty$. Observe that the matrix of coefficients

$$\begin{pmatrix} 1 & a & b \\ b & 1 & a \\ a & b & 1 \end{pmatrix}$$

can be chosen arbitrarily close to a symmetric matrix and still exhibits persistent oscillations. Therefore a Lyapunov function in the sense of Theorem 2.1 cannot exist for this system.

We replaced the symmetry assumption by assumptions (H3) and (H4). These two are obviously satisfied if the matrix W is symmetric. Observe that in the system (4) the condition (H4) is not satisfied; along the cycle $C := e_{12}, e_{23}, e_{31}$ we have

$$p(C) = \beta_{12}\beta_{23}\beta_{31} = a^3 \neq b^3 = \beta_{21}\beta_{32}\beta_{13} = \bar{p}(C).$$

Observe that the assumption (H4) is always satisfied in the absence of cycles, i.e. when the graph $G(W)$ is a tree.

In order to show that the system is quasi-convergent we need to show that forward trajectories are bounded. To this end, a natural assumption is the *point dissipativeness* assumption

$$\text{all trajectories eventually enter a positively invariant bounded set } B. \tag{D}$$

Remark 2.2. Point dissipativness is guaranteed, for example, by the following set of assumptions:

Assume that all f_i are bounded, $a_i(\mathbf{x}) > 0$ for sufficiently large $|\mathbf{x}|$ and $\gamma_i(x_i)x_i \to -\infty$ for $|x_i| \to \infty$. It is easy to see that under these assumptions $\dot{x}_i x_i < 0$, for every i, if $|x_i|$ is sufficiently large.

We also remark that in the context of neural networks it is customary to assume that the functions f_j are bounded.

Corollary 2.3. *Consider system (3) with (H1-H4) and (D). Then the system (3) is quasi-convergent. If the set of equilibria is finite, then the system is convergent.*

This result has an interesting consequence for Lotka-Volterra systems. Since in this case the physically meaningful region of the phase space is the nonnegative orthant $\mathbf{R}^{n+}$, the assumption (H2) is automatically satisfied.

Corollary 2.4. *Consider Lotka-Volterra system (2) on $\mathbf{R}^{n+}$ with the graph $G(A)$ of interactions being a tree. Assume (H1) and (H3). Then all bounded trajectories converge to the set of equilibria. Moreover, heteroclinic cycles are excluded, even on the boundary of $\mathbf{R}^{n+}$.*

To see that assumption (H3) is sharp, consider a two dimensional predator-prey Lotka-Volterra model

$$\begin{aligned}\dot{x}_1 &= x_1(1-x_2)\\ \dot{x}_2 &= x_2(-1+x_1).\end{aligned}$$

This system satisfies (H1) and the graph of interactions is a tree. The condition (H3) is not satisfied and, as is well known (see for instance Hale and Koçak [1991]), all solutions starting in the interior of the positive orthant are periodic.

3 Proofs

We first prove an abstract combinatorial result which may be of independent interest to researchers in combinatorial matrix theory. It generalizes theorems of Parter and Youngs [1962] and that of J. Maybee [1974].

Theorem 3.1. *Let W be real, combinatorially symmetric, irreducible matrix. Then there is a positive diagonal matrix D such that DW is a symmetric matrix if, and only if,*

1. *there is a spanning tree T of the graph $G(W)$ such that for every edge $e_{ij} \in T$ we have $w_{ij}w_{ji} > 0$ and*
2. *if C_T is a chord cycle in the graph G with respect to tree T, then*

$$p(C_T) = \bar{p}(C_T).$$

Observe, that 1. and 2. are precisely assumptions (H3) and (H4).

Proof. Assume 1. and 2. We seek to define numbers v_i, $i = 1, \dots n$ such that

$$\frac{w_{ij}}{w_{ji}} = exp(v_i - v_j). \tag{5}$$

To construct the collection $\{v_i\}_{i=1}^n$ we pick v_1 and compute v_i for all i with the property that $e_{1i} \in T$, using (5). In this way we construct all v_i and, obviously, (5) is satisfied along all the edges of T. Let us consider an edge $e_{kl} \in (G \setminus T)$. Since T is a spanning tree of $G(W)$, there is a path P in T from vertex k to vertex l. Thus $C = P \cup e_{kl}$ forms a chord cycle and by assumption 2., along this cycle we have $p(C) = \bar{p}(C)$, which we write as

$$\Pi_C \frac{w_{ij}}{w_{ji}} = 1.$$

Since (5) holds along edges in P this implies

$$\begin{aligned}1 &= \Pi_C w_{ij}/\Pi w_{ji}\\ &= \frac{w_{kl}}{w_{lk}}\Pi_P\, exp(v_i - v_j).\end{aligned}$$

Simplifying the right hand side we get

$$\frac{w_{kl}}{w_{lk}} = exp(v_k - v_l).$$

Since e_{kl} was arbitrary, collection $\{v_i\}$ satisfies (5) for all edges in $G(W)$.

Let $d_i := exp(-v_i)$ and let $D = diag(d_i)$. Let $Q = DW$. Observe that

$$\begin{aligned}q_{ij}/q_{ji} &= \frac{d_i}{d_j}\frac{w_{ij}}{w_{ji}}\\ &= exp(-v_i)exp(v_j)exp(v_i - v_j) = 1\end{aligned}$$

and so Q is a symmetric matrix.

If we assume that $Q = DW$ is symmetric, 1. and 2. follow easily. □

Proof of Theorem 2.1 Observe that (H3) and (H4) are assumptions 1. and 2. in Theorem 3.1. We assume without loss of generality that W is irreducible. In the opposite case we consider every irreducible component separately. We use the symmetric matrix $Q = [q_{ij}]$ and the diagonal matrix $D = [d_{ij}]$ from Theorem 3.1. Let

$$b_i := 2d_i \int^{x_i} (\frac{df_i}{d\zeta_i}\gamma_i)(\zeta_i)d\zeta_i.$$

We define the function $V : \mathbf{R}^n \to \mathbf{R}$

$$V(\mathbf{x}) := -\sum_{i=1}^{n}\left(b_i(x_i) - \sum_{j=1}^{n} q_{ij} f_i(x_i) f_j(x_j)\right). \tag{6}$$

We calculate the derivative of V, abbreviating $f_i' = \frac{df_i}{dx_i}$ to simplify notation.

$$\begin{aligned}\dot V &= -\sum_{i=1}^{n}\left(\frac{db_i}{dx_i}\dot x_i - \sum_{j=1}^{n} q_{ij}(f_i' f_j(x_j)\dot x_i + f_j' f_i(x_i)\dot x_j)\right)\\ &= -\sum_{i=1}^{n}\left(2d_i f_i' \gamma_i \dot x_i - \sum_{j=1}^{n}(q_{ij} f_i' f_j(x_j)\dot x_i + q_{ji} f_i' f_j(x_j)\dot x_i)\right) = (*)\end{aligned}$$

since, by rearranging the summations,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} f_i' f_j(x_j)\dot x_i + q_{ij} f_j' f_i(x_i)\dot x_j = \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} f_i' f_j(x_j)\dot x_i + q_{ji} f_i' f_j(x_j)\dot x_i.$$

We continue our computation

$$\begin{aligned}(*) &= -\sum_{i=1}^{n}\left(2d_i f_i' \dot x_i(\gamma_i - \sum_{j=1}^{n}\frac{q_{ij}+q_{ji}}{2d_i} f_j(x_j))\right)\\ &= -\sum_{i=1}^{n}\left(2d_i f_i' \dot x_i(\gamma_i - \sum_{j=1}^{n}\frac{2q_{ij}}{2d_i} f_j(x_j))\right)\\ &= -\sum_{i=1}^{n} 2d_i f_i' a_i(\mathbf{x})[\gamma_i - \sum_{j=1}^{n} w_{ij} f_j(x_j)]^2\end{aligned}$$

where we used equations (3) and that $w_{ij} = q_{ij}/d_i$. To finish the computation we observe that d_i are positive constants, $f_i' > 0$ by the assumption (H1), and $a_i(\mathbf{x}) \geq 0$ by assumption (H2). Thus

$$\dot V \leq 0.$$

Since $2d_i f_i'$ is strictly positive, $\dot V = 0$ if, and only if, $a_i(\mathbf{x})[\gamma_i - \sum_{j=1}^{n} w_{ij} f_j(x_j)]^2 = 0$ for every i. This is equivalent to

$$a_i(\mathbf{x})(\gamma_i - \sum_{j=1}^{n} w_{ij} f_j(x_j)) = 0$$

for every i and, consequently, $\dot x_i = 0$ for all i. Hence $\dot V(x) = 0$ if, and only if, x is an equilibrium. This finishes the proof of Theorem 2.1. □

It i interesting to note, that we do not know whether the result of J. Maybee [1974] (below), which we generalize in our Theorem 3.1, can be also used to construct a Lyapunov function V and prove Theorem 2.1.

Theorem 3.2 (Maybee (1974)). *Let W be a combinatorially symmetric, irreducible matrix. Then there is a real diagonal matrix D such that $Q := D^{-1}WD$ is symmetric if, and only if, (H3) and (H4) hold.*

We remark that from the proof of this Theorem in Maybee [1974] it is clear that if (H3) and (H4) hold then matrix D can be made to be a positive matrix.

Proof of Corollary 2.4 On the closed positive orthant $\mathbf{R}^{n+}$ the assumption (H2) is satisfied automatically. The graph of interactions $G(W)$ is a tree and there is a Lyapunov function V on the closed positive orthant by Theorem 2.1. Thus every bounded trajectory converges to the set of equilibria. Since the Lyapunov function is defined also on the boundary of the positive orthant we can exclude the existence of the heteroclinic cycles on the boundary. This proves the Corollary. □

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