

# Continuity of resetting curves for FitzHugh-Nagumo equations on the circle.

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## 1 Introduction

Abnormal cardiac rhythms are a serious cause for illness and death. In tachycardias, the heartbeat is too rapid. Even though tachycardias can arise in different regions of the heart, the underlying mechanisms of these rhythms may be similar. One mechanism for tachycardia assumes that the excitation follows a reentrant path. This means that the period of the oscillation is determined by the length of time it takes for an excitation to follow a circuitous path, rather than the period of the normal pacemaker of the heart. An early proposal by Mines was that reentrant cardiac activity could be modeled by excitation traveling in a ring of cardiac tissue – a viewpoint that has remarkably formed the basis for modern analyses of some arrhythmias (Josephson et al. [1993]).<sup>1</sup> The hypothesis that an excitation is traveling in a ring of tissue has an important implication to the cardiologist. If the ring is cut, the anatomical substrate for the rhythm is eliminated and the tachycardia does not reoccur. The development of ablation methods, that can often be employed to destroy abnormal cardiac tissue and cure the patient, provides a partial support for the theoretical model. Of course, the real heart is a complex three-dimensional structure and the cardiologist's notion of excitation traveling on a ring is really a topological notion. Thus, though they have usually not studied the mathematics, at least some cardiologists approach their field using a topological perspective.

The current paper is motivated by a set of numerical studies and conjectures generated by Glass and colleagues (Glass and Josephson [1995], Nomura and Glass [1996]). Reentrant tachycardia was modeled by excitation traveling on a one dimensional ring. The model had the following properties:

1. The system is an *excitable medium*. This means that it supports traveling waves, called excitation waves, similar to what is observed in nerve and heart. Following an excitation the tissue is refractory and it cannot support another excitation, for a length of time called the refractory time. As a consequence, two excitations in a one dimensional ring traveling towards each other in opposite senses will annihilate each other. One of accepted models of an excitable medium is described by the FitzHugh-Nagumo equations.

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<sup>1</sup>Reentrant excitation can also have other geometries such as spiral waves, scroll rings, or other complex topologies, but these notions will not occupy us here.

2. The ring is sufficiently large that a traveling wave will travel around the ring at a constant velocity.
3. A stimulus delivered to the ring at a fixed location of the ring far outside of the refractory time will lead to initiation of two waves traveling around the ring in opposite directions.

Glass and Josephson [1995] proposed the following conjecture:

**Conjecture 1.1** *Given an excitable medium in a one-dimensional ring supporting reentrant excitation, stimuli can be found which would lead to annihilation of the reentrant wave.*

The conjecture is consistent with cardiological practice. Cardiologists often deliver single or multiple stimuli to terminate reentrant tachycardia. Cardiologists often consider the ability to terminate a tachycardia by pacing a diagnostic feature of reentrant tachycardia (Josephson et al. [1993]). Finally, a class of medical devices (anti-tachycardia pacers) have been designed based on it (Saksena and Goldschlager [1990]).

The current paper was motivated by an attempt to prove the conjecture using following strategy. Precise definitions of unfamiliar terms will be presented below.

1. Excitable media in the geometry of a one-dimensional ring, support a stable circulating pulse, i.e., an asymptotically stable limit cycle in the associated partial differential equation.
2. Continuity Theorem: If a perturbation delivered at any phase of a limit cycle oscillation leaves the state point in the basin of attraction of the asymptotically stable limit cycle, then the resetting curves characterizing the effects of the stimuli, will be continuous.
3. Resetting curves for stably circulating pulses on a one dimensional ring are discontinuous.
4. It follows from 2. and 3. that there is a critical stimulus (or stimuli) that will lead to a point in the phase space outside of the basin of attraction of stable periodic orbit.

In ordinary differential equations, the continuity Theorem follows from arguments by Guckenheimer [1975]. The current manuscript extends the continuity Theorem to partial differential equations, in particular, to the FitzHugh-Nagumo equations on the ring. Proofs for 1. and 3. above have not yet been found, although an argument for 3. was presented in Glass and Josephson [1995]. Another goal of this paper is to provide theoretical framework for discussion of continuity of resetting curves in other models, for instance, using functional differential equations (Glass [1997]). This is the reason why we state and prove continuity theorem in a rather abstract setting.

The paper is organized as follows. In section 2, we define precisely resetting function for a semiflow on a complete metric space and prove continuity theorem. The assumptions of the theorem are abstract and the rest of the paper is devoted to verification of these assumptions in various settings. In section 3 we show that the assumptions of continuity theorem are verified for flows on finite dimensional spaces. In section 4 we deal with infinite dimensional problems. We briefly mention main differences with the finite dimensional problems and then, following Henry [1981], we verify the assumptions of continuity theorem for semilinear parabolic equations, where the linear operator is sectorial.

Finally, in section 5 we apply the results of section 4 to FitzHugh-Nagumo equations on a circle.

## 2 Continuity theorem

We start with a few definitions which lead to a concept of phase resetting function and continuity theorem.

Let  $X$  be a complete metric space and  $\mathbf{R}^+ = [0, \infty)$ . A family of mappings  $T(t) : X \rightarrow X$ , is said to be a  $C^r$ -semigroup,  $r \geq 0$ , provided that

- (i)  $T(0) = I$ , the identity map,
- (ii)  $T(t + s) = T(t)T(s)$ ,  $t \geq 0$ ,  $s \geq 0$ ,
- (iii)  $T(t)x$  is continuous in  $t, x$  together with Fréchet derivatives in  $x$  up through order  $r$  for  $(t, x) \in \mathbf{R}^+ \times X$ .

We shall also use notation  $\Phi(t, x) := T(t)x$  for the semiflow generated by the semigroup  $\{T(t)\}_{t \geq 0}$ .

**Definition 2.1** A compact invariant set  $S$  is a *local attractor* if there is an  $\delta$ -neighborhood  $B_\delta(S)$  of  $S$  such that

$$\text{dist}(\Phi(t, B_\delta(S)), S) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

**Definition 2.2** A compact invariant set  $S$  is a *local attractor with asymptotic phase* if it is a local attractor and there is a  $b$  such that for any  $x \in B_b(S)$  there is a  $y \in S$  such that

$$|\Phi(t, x) - \Phi(t, y)| \rightarrow 0 \text{ as } t \rightarrow \infty.$$

If  $S$  is a periodic orbit  $\Gamma = \{\gamma(t), 0 \leq t \leq p\}$ , this is equivalent to saying that there is a real number  $\Theta(x)$  such that

$$|\Phi(t + \Theta(x), x) - \gamma(t)| \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (1)$$

It follows from these definitions that if a periodic orbit  $\gamma$  is local attractor with asymptotic phase, and  $W$  is the stable manifold of  $\Gamma$ , there is a map  $\Theta : W \rightarrow S^1$  satisfying (1) for all  $x \in W$ .

**Definition 2.3** Let  $f : W \rightarrow W$  be a continuous map. The *phase resetting function*  $g_f : S^1 \rightarrow S^1$  is a function with the property

$$\Theta \circ f|_\Gamma = g_f \circ \Theta|_\Gamma,$$

where  $\Theta|_\Gamma$  and  $f|_\Gamma$  are restrictions of the corresponding functions to periodic orbit  $\Gamma$ .

Observe that the function  $\Theta$ , restricted to periodic orbit  $\Gamma$ , is a bijection from  $\Gamma$  to  $S^1$ . Therefore there is an inverse  $\Theta_\Gamma^{-1} : S^1 \rightarrow \Gamma$  and  $g_f = \Theta \circ f|_\Gamma \circ \Theta_\Gamma^{-1}$ . The function  $\Theta_\Gamma^{-1}$  maps an angle  $\theta \in S^1$  to a point  $\gamma(\theta) \in \Gamma$ . The perturbation function  $f$  maps  $\gamma(\theta)$  to a point  $f(\gamma(\theta)) \in W$ . Finally,  $\Theta(f(\gamma(\theta)))$  is the asymptotic phase shift (see (1)) of the trajectory starting at  $f(\gamma(\theta))$ .

**Theorem 2.4 (Continuity Theorem)** *Let  $X$  be a complete metric space with a semiflow  $\Phi(t, x)$ . Let  $\Gamma$  be a periodic orbit of the semiflow  $\Phi(t, x)$ , which is a local attractor with asymptotic phase. Let  $\Theta(x)$  be defined as above for  $x \in W$ , the stable manifold of  $\Gamma$ . Assume  $\Theta(x)$  is continuous.*

*Then the phase resetting function  $g_f$  is continuous for every continuous function  $f : W \rightarrow W$ .*

*Proof.* Observe,  $\Theta|_\Gamma$  is a bijection between  $\Gamma$  and  $S^1$ . Thus this map has a continuous inverse  $\Theta_\Gamma^{-1}$ . Therefore

$$g_f = \Theta \circ f|_\Gamma \circ \Theta_\Gamma^{-1} = \Theta \circ f \circ \Theta_\Gamma^{-1}.$$

Since all maps on the right hand side are continuous by assumption,  $g_f$  is continuous.  $\square$

### 3 Finite dimension

In this section we verify the assumptions of Theorem 2.4 for finite dimensional case. These results are well known in the literature.

**Theorem 3.1 (Theorem VI.2.1, Hale (1980))** *Consider  $\dot{x} = f(x)$  an ordinary differential equation on a finite dimensional manifold. Let  $\Gamma$  be a hyperbolic periodic orbit, which has simple multiplier one and all other multipliers have modulus less than one.*

*Then  $\Gamma$  is a local attractor with asymptotic phase.*

If the function  $\Theta(x)$  exists, then the stable manifold  $W$  is divided into sets  $\Theta^{-1}(\theta), \theta \in [0, p]$ , where  $p$  is the period of  $\gamma$ . If these sets vary continuously throughout  $W$ , then  $\Theta(x)$  will be continuous. This idea is captured in the following definition.

**Definition 3.2** A surjection  $\pi : E \rightarrow B$  between topological spaces is a *fiber bundle*, if  $B$  has an open cover  $\{U_\alpha\}$  such that there are fiber preserving homeomorphisms

$$\phi_\alpha : E|_{U_\alpha} \rightarrow U_\alpha \times F$$

and the transition functions

$$g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}|_{\{x\}} \times F$$

are continuous. The set  $\pi^{-1}(\{x\})$  is called *fiber at  $x$* .

In our problem we have  $E = W$ ,  $B = \Gamma$  and  $\pi = \Theta$ .

**Theorem 3.3 (Theorem A, Guckenheimer (1975))** *Let  $\Gamma$  be a periodic orbit of a flow on a finite dimensional manifold, which is local attractor with asymptotic phase. Then the stable manifold  $W$  of  $\Gamma$  is a fiber bundle  $\pi : W \rightarrow S^1$ .*

These two Theorems verify assumptions of Theorem 2.4. Therefore we have

**Corollary 3.4** *Consider  $\dot{x} = f(x)$  an ordinary differential equation on a finite dimensional manifold. Let  $\Gamma$  be a hyperbolic periodic orbit, which has a simple multiplier one and all other multipliers have modulus less than one.*

*Then the phase resetting function  $g_f$  is continuous for every continuous function  $f : W \rightarrow W$ .*

## 4 Infinite dimensional problems

We begin by pointing out the differences from the finite dimensional case and then proceed with a particular case of FitzHugh-Nagumo equation on the circle.

Consider

$$u_t = -Au + f(u)$$

with  $u(0) = u_0 \in X$ , a Banach space. Here  $A$  is a linear operator and  $f(u)$  is a nonlinearity. A book by Hale [1988] is a great source for results about infinite dimensional problems. We follow section 3.10 to discuss two major problems with Poincaré maps for periodic orbits in infinite dimensional problems.

The first problem is that in order to define a Poincaré section, periodic orbit  $\Gamma = \{\gamma(t) \mid 0 \leq t \leq p\}$  must be  $C^1$ . This is true, provided  $\Gamma$  lies on the global attractor (see Hale[1988]) and  $|f'(u)|$  is bounded in the neighborhood of  $\Gamma$ .

The second problem is the smoothness of the Poincaré map. Given a section  $S$  of the semiflow  $\Phi(t, x)$ , one defines Poincaré map on  $S$  by  $\Phi(t_r(x), x)$ , where  $t_r(x)$  is return time to  $S$  for the trajectory starting at  $x$ .

However, the map  $x \rightarrow t_r(x)$  is not necessarily smooth and thus a Poincaré map need not be smooth. Thus we cannot express stability of a periodic orbit in terms of the eigenvalues of the derivative of the Poincaré map. Instead, we consider a linear variational equation about  $\gamma(t)$

$$v_t = -Av + f'(\gamma(t))v. \tag{2}$$

Let  $U(t, \sigma)$  be a solution operator of (2). Then the characteristic multipliers of (2) are the elements of the point spectrum of  $U(\sigma + p, \sigma)$ , where  $p$  is the period of  $\gamma(t)$ . If we assume that the radius of the essential spectrum is  $< 1$  and  $\Gamma$  is a  $C^1$ -manifold then one is always a multiplier, since  $\dot{\gamma}(t)$  is a solution of (2).

**Definition 4.1** If one is a simple multiplier of (2) and no other multiplier has modulus one, we say that  $\Gamma$  is hyperbolic.

### 4.1 Semilinear parabolic equations

In this subsection we will follow work of D. Henry [1981] on semilinear parabolic equations. We shall verify the assumptions of the continuity Theorem in this class of equations, provided that the linear operator is sectorial. As in finite dimensional case we need two results. One, that the hyperbolic attracting periodic orbit is a local attractor with asymptotic phase, and second, that the associated function  $\Theta(x)$  is continuous. The first result can be found in Henry [1981]. The second result essentially follows from his proof, but we need to add some details.

**Definition 4.2 (Henry (1981))** A linear operator  $A$  in a Banach space  $X$  is *sectorial*, if it is a closed densely defined operator such that, for  $\varphi \in (0, \pi/2)$  and some  $M \geq 1$  and a real  $a$ , the sector

$$S_{a, \varphi} = \{\lambda \mid \varphi \leq |\arg(\lambda - a)| \leq \pi, \lambda \neq a\} \tag{3}$$

is in the resolvent set of  $A$  and

$$\|(\lambda - A)^{-1}\| \leq M/|\lambda - a| \text{ for all } \lambda \in S_{a, \varphi}. \tag{4}$$

A sectorial operator is a generalization of an elliptic operator  $\Delta$ . Given a Banach space  $X$  and a sectorial operator  $A$  one can define fractional powers  $A^\alpha$  of the operator  $A$  and for every  $\alpha \geq 0$  one can define corresponding fractional power spaces  $X^\alpha$ . It can be shown (Theorem 1.4.8 Henry [1981]) that  $X^0 = X$  and for  $\alpha \geq \beta \geq 0$ ,  $X^\alpha$  is a dense subset of  $X^\beta$  with continuous inclusion.

We consider

$$u_t = -Au + f(u) \tag{5}$$

and assume:

**Assumptions 4.3** Operator  $A$  is sectorial on a Banach space  $X$  and  $f$  is continuously differentiable from  $X^\alpha$  to  $X$ . Assume that  $\gamma(t)$  is a nonconstant periodic solution of period  $p$  of (5) and that  $t \rightarrow f_x(\gamma(t)) \in L(X^\alpha, X)$  is Hölder continuous.

**Lemma 4.4 (Lemma 8.2.2, Henry (1981))** *Assume (4.3). Then the linear variational equation (2) about  $\gamma(t)$  has one as a characteristic multiplier.*

**Theorem 4.5 (Theorem 8.2.3, Henry (1981))** *Assume (4.3) and that one is an isolated simple eigenvalue of the period map, and that the remainder of the spectrum lies in  $\{|\mu| < e^{-\beta p}\}$  for some  $\beta > 0$ .*

*Then  $\Gamma$  is a local attractor with asymptotic phase.*

**Remark 4.6** From the proof of Theorem 4.5 it is clear that  $\Theta(x)$  is well defined in a small neighborhood of  $\Gamma$  since only there one can guarantee existence of a solution for a sufficiently large time. Although the function  $\Theta(x)$  is defined on  $W$  since every solution converging to  $\Gamma$  must pass through this small neighborhood of  $\Gamma$ ,  $W$  does not have to be a “nice” neighborhood of  $\Gamma$ . Thus the assumption that the perturbation function  $f$  in the continuity theorem maps  $W$  to  $W$  is crucial and may be difficult to verify outside of a small neighborhood of  $\Gamma$ .

In order to verify the assumptions of the continuity theorem, it remains to show that the function  $\Theta(x)$  (which exists because of (4.5)) is continuous.

**Theorem 4.7** *Assume (4.3), that one is an isolated simple eigenvalue of the period map, and that the remainder of the spectrum lies in  $\{|\mu| < e^{-\beta p}\}$  for some  $\beta > 0$ . Then the function  $\Theta(x)$  is continuous on  $W$ .*

*Proof.* The proof of the Theorem follows from the proof of Theorem 4.5 in Henry [1981]. Since it is not explicitly spelled out we lead the reader through his argument and show how it can be changed to yield the proof of Theorem 4.7.

As a preliminary step Henry [1981] constructs a family of functions  $z(t, a), t \in [0, \infty)$ , depending on a parameter  $a$ , which related to an initial value of  $z(t, a)$  at  $t = t_0$ . These functions are such that  $x^*(t, a) = \gamma(t) + z(t, a)$  are solutions of (5) for  $t > t_0$  and converge to  $\gamma(t)$  exponentially.

Then he considers solution  $x(t, \zeta)$  of (5) such that

$$x(t_0 - p, \zeta) = \zeta. \tag{6}$$

In order to show convergence with asymptotic phase it is enough to show that there is  $\theta^*$  real and  $a^*$  such that

$$x(t_0 + \theta^*, \zeta) = x^*(t_0, a^*). \tag{7}$$

Indeed, uniqueness implies that  $x(t, \zeta) = x^*(t - \theta^*, a^*)$  and by construction of  $x^*(t, a^*)$  the later function converges exponentially to  $\gamma(t - \theta^*)$ .

To show that (7) has a solution, we must confine ourself to a small neighborhood of  $\Gamma$  to ensure that  $x(t, \zeta)$  exist for  $t_0 - p \leq t \leq t_0 + p$ .

Then we can recast solvability of (7) in terms of a operator equation

$$a - \theta \dot{\gamma}(t_0) = G(\theta, a; \zeta) \tag{8}$$

where  $G(0, 0; \gamma(t_0)) = 0$ . (For details one should consult Henry [1981].) The equation (8) has a solution  $(\theta^*, a^*)$  if and only if (7) has.

The left hand side is linear in  $\theta$  and  $a$  and thus can be thought of as a linear operator  $L$  applied to pair  $(\theta, a)$ . Henry shows that this operator is invertible, however no bound on the norm of the inverse  $\|L^{-1}\|$  is given. In other words we try to solve

$$(\theta, a) = L^{-1}G(\theta, a; \zeta)$$

where  $L$  is the aforementioned linear operator. Estimates in Henry [1981] show that the Lipschitz constant of  $G$  can be made arbitrarily small in a neighborhood of  $\Gamma$ . Thus  $L^{-1}G$  is a contraction and its unique fixed point  $(\theta^*, a^*)$  solves (7).

We want to focus on another aspect of (8) and that is that it represents a family of equations parameterized by the initial condition  $\zeta$ . Furthermore, the estimate of the Lipschitz constant does not depend on  $\zeta$  i.e. it is uniform in  $\zeta$ . Thus one can apply Contraction mapping theorem with parameters (Hale [1980]) to conclude that the function

$$\zeta \rightarrow (\theta^*, a^*)$$

is continuous. This map is defined so far for  $\|\zeta - \gamma(t_0)\|$  sufficiently small.

However, we can replace (6) with  $x(t_0 - T, \zeta) = \zeta$  for any  $T \in [0, p]$ , repeat the argument above and conclude the same result. Thus one can define a continuous function  $F$  defined on a sufficiently small neighborhood  $N$  of  $\Gamma$

$$F : \zeta \rightarrow (\theta^*, a^*).$$

Observe that for any  $\zeta, \eta \in W$  there is a finite time  $\tau$  such that both  $x(\tau, \zeta)$  and  $x(\tau, \eta)$  are in  $N$ . Thus we can extend  $F$  to  $W$ , and, by the continuous dependence on initial conditions,  $F$  is continuous on  $W$ .

Projection of continuous function  $F$  to the first variable is the function  $\Theta(\zeta)$ , which is therefore continuous.  $\square$

The only place in the above argument where the sectoriality of  $A$  is used is in the estimates which show that operator  $G$  has Lipschitz constant uniformly and arbitrarily small.

Since Theorem 4.5 and Theorem 4.7 verify assumptions of Theorem 2.4 we have:

**Corollary 4.8** *Consider  $u_t = -Au + f(u)$  where  $A$  is a sectorial operator on a Banach space  $X$  and  $f$  is continuously differentiable from  $X^\alpha$  to  $X$ . Let  $\gamma(t)$  be a nonconstant periodic solution, such that one is an isolated simple eigenvalue of the period map, and that the remainder of the spectrum lies in  $\{|\mu| < e^{-\beta p}\}$  for some  $\beta > 0$ . Assume that  $t \rightarrow f_x(\gamma(t)) \in L(X^\alpha, X)$  is Hölder continuous.*

*Then the phase resetting function  $g_f$  is continuous for every continuous perturbation function  $f : W \rightarrow W$ .*

## 5 FitzHugh-Nagumo equation on a circle

As a model for the reentrant wave on a circle of excitable media we consider FitzHugh-Nagumo equations on interval  $[0, 2\pi]$  with periodic boundary conditions (Glass and Josephson [1995], Glass and Nomura [1996]). Consider

$$\begin{aligned} u_t &= Du_{xx} + f(u) - w + I \\ w_t &= c(u - dw) \end{aligned} \tag{9}$$

on a space  $X := H_p^2(0, 2\pi) \times L_p^2(0, 2\pi)$  with periodic boundary conditions

$$u(0, t) = u(2\pi, t), u'(0, t) = u'(2\pi, t), w(0, t) = w(2\pi, t).$$

Function  $f \in C^2(\mathbf{R}, \mathbf{R})$  and  $c$  and  $d$  are constants. The norm on the space  $H$  is  $\|(u, w)\| = \|u\|_2 + \|w\|$  where the first norm is the norm in  $H_p^2(0, 2\pi)$  and the second in  $L_p^2(0, 2\pi)$ .

We note that  $H_p^2(0, 2\pi)$  is a Hilbert space,  $H_p^2(0, 2\pi)$  is a dense subset of  $L_p^2(0, 2\pi)$  and it can be characterized by (see Temam [1983])

$$H_p^2(0, 2\pi) = \left\{ u \mid u = \sum_{n \geq 0} a_n \sin(nx) + \sum_{n \geq 0} b_n \cos(nx), \sum_{n \geq 0} n^4 (|a_n|^2 + |b_n|^2) < \infty \right\}. \tag{10}$$

In agreement with the outline in the Introduction we assume that system (9) admits an hyperbolic periodic orbit, one is an isolated simple eigenvalue of the period map, and that the remainder of the spectrum lies in  $\{|\mu| < e^{-\beta p}\}$  for some  $\beta > 0$ .

The goal of this subsection is to verify the rest of the assumptions of Corollary 4.8. This shows that the resetting function is continuous, provided that the perturbation function  $f$  maps  $\Gamma$  into  $W$ . We first show that the linear operator associated with (9) is sectorial.

We rewrite (9) as

$$v_t = -Av + F(v)$$

where  $v = (u, w)^T$ ,  $F(v) = (f(u) - \frac{\partial f}{\partial u}|_{u=0}, 0)$  is the nonlinear part, and

$$A = \begin{bmatrix} -D \frac{\partial^2}{(\partial x)^2} - \frac{\partial f}{\partial u}|_{u=0} & 1 \\ -c & cd \end{bmatrix}.$$

We compute the spectrum of  $A$ . The spectrum  $\sigma(A) = \sigma_p(A) \cup \sigma_r(A) \cup \sigma_c(A)$  decomposes (Naylor and Sell [1982]) to point spectrum  $\sigma_p(A)$ , residual spectrum  $\sigma_r(A)$  and continuous spectrum  $\sigma_c(A)$ .

First we consider a point spectrum  $\sigma_p(A)$ . Note that the collection

$$\left\{ \frac{1}{\sqrt{\pi}} \sin(nx), \frac{1}{\sqrt{\pi}} \cos(nx) \right\}_{n=1}^{\infty},$$

together with constant functions, forms an orthonormal basis of  $L_p^2(0, 2\pi)$ . We set  $v = (a_n^1, a_n^2)^T \frac{1}{\sqrt{\pi}} \sin(nx)$  where  $(\cdot)^T$  denotes the transpose vector. We compute the eigenvalues from

$$(A - \lambda I)v = 0.$$

We shall omit analogous argument with  $v = (b_n^1, b_n^2)^T \frac{1}{\sqrt{\pi}} \cos(nx)$ . The characteristic equation is

$$\lambda^2 - \lambda(cd + Q) + Qcd + c = 0 \tag{11}$$

where  $Q = Dn^2 - \frac{\partial f}{\partial u}|_{u=0}$ .

Equation (11) has complex roots if and only if

$$(cd - Q)^2 - 4c < 0. \tag{12}$$

Since  $c, d$  and  $\frac{\partial f}{\partial u}|_{u=0}$  are constants and

$$(cd - Q)^2 \rightarrow \infty \text{ as } n \rightarrow \infty$$

there is only finitely many  $n$  for which (12) is satisfied. Therefore there is only finitely many eigenvalues with nonzero complex part and it follows that there is a sector  $S_{a,\varphi}$  (see (3)) such that  $\mathbf{C} \setminus S_{a,\varphi}$  contains point spectrum  $\sigma_p(A)$ .

We need to study continuous and residual spectrum of  $A : X \rightarrow L := L_p^2(0, 2\pi) \times L_p^2(0, 2\pi)$ .

To that end we study properties of the resolvent operator  $(A - \lambda I)^{-1}$  on  $L$ . Consider

$$(A - \lambda I)v = F \tag{13}$$

for  $F = (f_1, f_2) \in L$  and  $v = (u, w)$ . Then the second equation in (13) reads

$$-cu + cdw = f_2 + \lambda w$$

which can be solved for  $w$

$$w = -\frac{cu + f_2}{\lambda - cd}$$

if and only if

$$\lambda \neq cd. \tag{14}$$

The first equation in (13) now yields

$$-\frac{\partial^2 u}{(\partial x)^2} - au - \lambda u - \frac{c}{\lambda - cd} u = f_1 + f_2/(\lambda - cd)$$

where  $a = \frac{\partial f}{\partial u}|_{u=0}$ . Observe that the right hand side,  $h := f_1 + f_2/(\lambda - cd) \in L_p^2(0, 2\pi)$ .

We solve for  $u$  by setting

$$u = \sum_{n \geq 0} a_n \sin(nx) + \sum_{n \geq 0} b_n \cos(nx)$$

and  $h = \sum_{n \geq 0} c_n \sin(nx) + \sum_{n \geq 0} d_n \cos(nx)$ . We only show the calculation for  $\sin(nx)$  modes since the calculation for  $\cos(nx)$  modes is analogous. The equation for the  $n$ -th mode is

$$(n^2 - a - \lambda - \frac{c}{\lambda - cd})a_n = c_n$$

and so

$$a_n = c_n / (n^2 - a - \lambda - \frac{c}{\lambda - cd}). \quad (15)$$

Analogously, for  $\cos(nx)$  modes we have

$$b_n = d_n / (n^2 - a - \lambda - \frac{c}{\lambda - cd}). \quad (16)$$

Obviously,  $a_n$  and  $b_n$  are not defined when the denominator vanishes. However, an easy calculation shows that  $n^2 - a - \lambda - \frac{c}{\lambda - cd} = 0$  if and only if  $\lambda \in \sigma_p(A)$ , the point spectrum of  $A$ . If  $a_n$  and  $b_n$  are defined then

$$\sum_{n \geq 0} n^4 (|a_n|^2 + |b_n|^2) < \infty$$

and so  $u \in H_p^2(0, 2\pi)$  by (10). This implies, that if  $\lambda \in \Lambda := C \setminus \{\sigma_p(A) \cup \{cd\}\}$ , then

$$(A - \lambda I)^{-1} : L \rightarrow X.$$

Since  $X$  is a dense subset of  $L$ , the set  $\Lambda$  does not intersect the residual spectrum  $\Lambda \cap \sigma_r(A) = \emptyset$ .

For every fixed  $\lambda \in \Lambda$  let  $k_\lambda$  be the minimum distance of  $\lambda$  to  $\sigma_p(A)$  in the complex plane. Then the norm of the operator  $(A - \lambda I)^{-1}$  is given by

$$\sup \left\{ \sum_{n \geq 0} |a_n|^2 + |b_n|^2 \right\}$$

where  $a_n$  and  $b_n$  satisfy (15) and (16) and the supremum is taken over all sequences  $c_n, d_n$  with  $\sum_{n \geq 0} |c_n|^2 + |d_n|^2 = 1$ . It is easy to see that

$$\sum_{n \geq 0} |a_n|^2 + |b_n|^2 \leq \frac{1}{k_\lambda} \left( \sum_{n \geq 0} |c_n|^2 + |d_n|^2 \right) = \frac{1}{k_\lambda}. \quad (17)$$

Hence  $(A - \lambda I)^{-1}$  is continuous and the set  $\Lambda$  does not intersect the continuous spectrum  $\Lambda \cap \sigma_c(A) = \emptyset$ .

A short calculation shows that if  $\lambda = cd$  then

$$(A - \lambda I)^{-1}(f_1, f_2) = (-f_2/c, f_1 + f_2/c(\lambda + a) - \frac{\partial^2 f_2}{(dx)^2}/c).$$

Observe that this operator is defined for  $(f_1, f_2) \in L_p^2(0, 2\pi) \times H_p^2(0, 2\pi)$  which is a dense subset of  $L$ . However, as we can find  $f_2 \in H_p^2(0, 2\pi)$  with the norm  $\|\frac{\partial^2 f_2}{(dx)^2}\|_2$  arbitrarily large, the operator  $(A - \lambda I)^{-1}$  is not bounded. Hence  $\lambda = cd$  is in the continuous spectrum  $\sigma_c(A)$ .

This shows that spectrum of  $A$ ,  $\sigma(A) = \sigma_p(A) \cup \{cd\}$ . Since  $cd$  is real, the set  $\mathbf{C} \setminus S_{a,\varphi}$  which contains point spectrum  $\sigma_p(A)$ , also contains  $cd$  and thus  $\sigma(A)$ . Therefore the sector  $S_{a,\varphi}$  is contained in the resolvent set of  $A$ . The equation (17) verifies (4) and so  $A : X \rightarrow L$  is a sectorial operator.

To verify the remaining assumptions of Corollary 4.8, we need to show that the nonlinearity

$$F := (f_1(u), 0) = (f(u) - \frac{\partial f}{\partial u}|_{u=0}, 0)$$



is continuously differentiable,  $F : X^\alpha \rightarrow X$ , for some  $\alpha$ . We first identify  $X^\alpha$ . Recall, (Henry [1981]) that if  $u = \sum c_i e_i$  is the expansion of  $u$  in basis  $e_i$  of  $X$ , then  $Au = \sum \lambda_i c_i e_i$ , where  $\lambda_i$  are eigenvalues of  $A$ . We define fractional power of  $A$  by  $A^\alpha u = \sum \lambda_i^\alpha c_i e_i$ . The space  $X^\alpha$  is the domain of  $A^\alpha$ ,  $X^\alpha = D(A^\alpha)$ , with the graph norm  $\|x\|_\alpha = \|A^\alpha x\|$ ,  $x \in X^\alpha$ .

It follows from (11) and (12) that

$$\operatorname{Re}(\lambda_n) = O(n^2) \text{ as } n \rightarrow \infty. \quad (18)$$

Using (10) we see that

$$X = \{u | u = \sum_{n \geq 0} (a_n^1, a_n^2)^T \sin(nx) + \sum_{n \geq 0} (b_n^1, b_n^2)^T \cos(nx), \sum_{n \geq 0} n^4 (|a_n^1|^2 + |b_n^1|^2) < \infty, \sum_{n \geq 0} |a_n^2|^2 + |b_n^2|^2 < \infty\}$$

and, by definition of  $X^\alpha$  and (18)

$$X^\alpha = \{u | u = \sum_{n \geq 0} (a_n^1, a_n^2)^T \sin(nx) + \sum_{n \geq 0} (b_n^1, b_n^2)^T \cos(nx), \sum_{n \geq 0} n^{4\alpha} (|a_n^1|^2 + |b_n^1|^2) < \infty, \sum_{n \geq 0} |a_n^2|^2 + |b_n^2|^2 < \infty\}.$$

From the embedding theorems (Henry [1981]) follows that if  $\alpha > 1/4$  then  $X^\alpha \subset C(0, 2\pi) \times L^2(0, 2\pi)$ . Since the nonlinearity  $F$  involves only function of the first component  $u$ , which is a continuous function, it is easy to see that  $F : X^\alpha \rightarrow X$  is continuously differentiable for any  $\alpha > 1/4$ . Choose  $\alpha = 1/2$ . It follows now that the problem (9) is locally well posed in  $X^{1/2}$ . To show that the solutions exist in  $X^{1/2}$  for all time, we use Lyapunov function  $V = \frac{1}{2} \|A^{1/2} u\|^2 + \int_0^{2\pi} F(u)$ . From the standard argument (see Hale [1988] section 4.3.), it follows that solutions of (9) form a semigroup, which is analytic by Theorem 1.3.4 of Henry [1981]. Therefore, since  $t \rightarrow \gamma(t)$  is analytic and  $\gamma(t) \rightarrow F_x(\gamma(t))$  is continuous,  $t \rightarrow F_x(\gamma(t))$  is continuous.

This verifies assumptions of Corollary 4.8.

## 6 Discussion

The goal of this paper was to prove continuity theorem. It is one of the ingredients in the proof of a general conjecture (Glass and Josephson [1995], Nomura and Glass [1996]) that there is always a stimulus which leads to annihilation of the wave of excitation, traveling around a ring of excitable media. We proved continuity theorem in a general setting of semiflow on a complete metric space. Thus the theorem is applicable to a wide variety of models, ranging from ordinary differential equations, partial differential equations to functional differential equations.

We pay the price for the simplicity and generality of the theorem and its proof when we verify the assumptions of the theorem in the concrete setting of FitzHugh-Nagumo equations on the circle. Since the phase space for the semiflow, generated by these equations, is infinite dimensional, one must work harder in order to avoid problems mentioned in section 4. Though calculations in section 5 are tedious, they are straightforward and commonly used in literature (Hale [1988]).

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