Cyclic feedback systems.

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1 Introduction

Study of dynamical systems usually concentrates on the properties and the structure of invariant sets, since the understanding of these is the first step in describing the long time behavior of orbits of the entire dynamical system.

There are two different sets of problems related to study of dynamical systems. One, the study of the dynamics in the neighborhood of the critical elements like fixed points or periodic orbits, is relatively well understood and the main tool used in these problems is the linearization and the bifurcation analysis. These problems are local in nature; assertions say something about the small neighborhood of the critical set in the phase space or a small neighborhood of the critical value in the parameter space.

The second set of problems is related to a global dynamics and the global bifurcations. These address issues of the existence of heteroclinic or homoclinic orbits and related bifurcation problems. There are few tools available to tackle these problems. Some, as a singular perturbation, convert the global problem in phase space into a local problem in the parameter space. Other methods use the topology of the phase space to show the existence of heteroclinic orbits. We mention mountain-pass type theorems for variational problems and the Conley index theory as examples of these methods.

Taking into account all the difficulties connected with the global dynamics, it is remarkable, that there are classes of equations, where the description of dynamics is possible. Even more so, if the phase space is $\mathbf{R}^n$ for $n > 2$.

In this work we study such a class of ordinary differential equations, called *cyclic feedback systems*. Though the results, which we shall describe, are important for the models, which use the equations from this class, we think, that at least as important is the type of questions we pose and type of answers we expect.

A system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_n) \\ \dot{x}_2 &= f_2(x_2, x_1) \\ &\vdots \\ \dot{x}_n &= f_n(x_n, x_{n-1}) \end{aligned} \tag{1}$$

is called a cyclic feedback system ($\mathcal{CFS}$) if there exist constants $\delta_i \in \{\pm 1\}$, $i = 1, \ldots, n$ such that

$$\begin{aligned} &\delta_i f_i(0, v)v > 0 \qquad \text{if } v \neq 0 \\ &\delta_i \frac{\partial f_i(a,b)}{\partial b}|(0,0) > 0 \end{aligned} \tag{2}$$

holds for $i = 1, \ldots, n$.

The word cyclic comes from the fact, that we identify $x_{n+1} = x_1$ and so the variables can be thought of as if they were arranged in a cyclic fashion. The constants δ_i determine the sign of a feedback; $\delta_i = 1$ has the interpretation of positive feedback since if $x_i = 0$ and $x_{i-1} \neq 0$ then $x_{i-1}\dot{x}_i > 0$ and $\delta_i = -1$ of negative feedback, since if $x_i = 0$ and $x_{i-1} \neq 0$ then

$x_{i-1}\dot{x}_i > 0$. There are two types of these systems characterized by the quantity $\Delta = \delta_1\delta_2\ldots\delta_n$ since, by the change of variables,

$$\begin{aligned} x_1 &\mapsto x_1 \\ x_i &\mapsto \delta_2\delta_3\ldots\delta_i x_i, \qquad i = 2,\ldots,n \end{aligned}$$

we get a system where $\delta_i = 1$ for $i \geq 2$ and

$$sgn\ \delta_1 = sgn\ \Delta = \pm 1$$

A negative feedback is used to stabilize a given system to the desired behavior in the control theory. In engineering the interest in feedback systems is focused on the stabilizing properties of the (negative) feedback i.e. to the range (and a boundary) of a region of parameters, where the system has a unique stable equilibria.

Apart from engineering, models, which belong to the class of cyclic feedback systems, have been used to model various biological phenomena. One of the first to suggest the importance of the feedback in biological systems were Jacob and Monod [15], who modeled genetic regulatory mechanism in bacteria using feedback systems. As an example of other models, we mention the work of Morales and McKay [21], who used a cyclic feedback system to model metabolic pathways in a cell and the work of Weiss and Kavanau [26] who modeled control mechanisms of a growth of cells. For a more comprehensive list of models we refer the reader to the paper of Hastings et.al. [9].

It should be noted that in most of the above mentioned papers the stabilizing effect of a negative feedback was explored. However, we would like to suggest that cyclic feedback systems are well suited for modeling and exploring oscillatory phenomena, because a large class of cyclic feedback systems exhibit a persistent periodic behavior. This phenomena is well known to engineers, who know the danger of "overcontroling " the system by imposing a too large negative feedback. This results in sustained oscillations, which are certainly not welcome in control theory, but may be precisely an aspect biologists want to explore.

As an illustration of a model, which uses a cyclic feedback system to model oscillatory phenomena, we mention the work of Atyia and Baldi [1] (see also references therein). They used a cyclic feedback system to study a simple neural network with a "ring architecture". The neurons form a ring and every neuron is connected to one neighbor in a cyclic fashion.

This architecture has received increased attention in the recent years because of its ability to support stable periodic orbits, which may be viewed as a stored spatio-temporal information.

There is a subclass of the class of cyclic feedback systems, called *monotone cyclic feedback systems* ($\mathcal{MCFS}$), which were studied extensively in the past, because of their ties to *monotone dynamical systems*. For $\mathcal{MCFS}$ conditions (2) are replaced by the stronger condition

$$\delta_i \frac{\partial f_i(a,b)}{\partial b} > 0 \quad i = 1,\ldots,n \tag{3}$$

i.e. the functions f_i are monotone in the second variable. If $\Delta = 1$ in a $\mathcal{MCFS}$, then the flow, given by (1) with (3), generates a monotone dynamical system. These systems have been studied by many authors, for instance by H. Matano [20], M. Hirsch [11, 12, 13, 14], H.

Smith [24] and others. An important property of these systems is that almost all trajectories converge to a fixed point (Hirsch [12, 13]).

In analyzing the systems of differential equations one often starts with local linear analysis around equilibria and very often, mainly if the system is high dimensional, this is where the rigorous treatment stops. The reason is simple: the global analysis of the high dimensional systems of ordinary differential equations is very difficult. Therefore it is always remarkable to find a class of equations, where one can say something more about the overall structure of the solutions. Cyclic feedback systems form one such class.

When we view the system of ordinary differential equations as a dynamical system, the emphasis is on the long term behavior of (most of) the orbits. This leads to a notion of a global attractor, where all the interesting dynamics takes place. We denote by $\omega(x)$ the *ω-limit set* of $x \in \mathbf{R}^n$,

$$\omega(x) := \bigcap_{t \geq 0} \overline{\varphi(x, [t, \infty))}$$

where $\varphi(x,t)$ denotes the flow generated by (1) and $\bar{A}$ is the closure of A. An *α-limit set* of the point $x \in \mathbf{R}^n$ is the set

$$\alpha(x) := \bigcap_{t \leq 0} \overline{\varphi(x, [t, \infty))}.$$

A set S is an *invariant set* of the flow $\varphi(x,t)$ if $\varphi(S,t) = S$ for every t. A set $\mathcal{A}$ is a *global attractor* if $\mathcal{A}$ is compact, invariant, and $\omega(B) \subset \mathcal{A}$ for every bounded set $B \subset \mathbf{R}^n$. Here $\omega(B) := \bigcap_{t \geq 0} \overline{\varphi(B, [t, \infty))}$.

In this paper we assume the existence of the global attractor for every system (1) we consider and attempt to describe the dynamics on this global attractor. This assumption can be translated to a condition on the growth of the nonlinearities at ∞ for every concrete model in hand; however, to avoid unnecessary restrictions, we shall consider the class given by (1) together with this general assumption. Since this class is quite large, we certainly shall not describe the dynamics on the attractor precisely for every possible nonlinearity and every possible n, but we shall exhibit certain common features of all these attractors. We now proceed to describe the two most important tools in our analysis:

- a discrete Ljapunov function
- a Morse decomposition.

A discrete Ljapunov function is a continuous function with values in the set of integers $\mathbf{Z}$ with the property that it is non-increasing along the trajectories. It is this last property which justifies the name Ljapunov function. There are notable differences between a real valued and a discrete valued Ljapunov function. A real valued Ljapunov functions naturally occur in dissipative systems, where the total energy decreases with time. The energy function serves then as a Ljapunov function. If the continuous Ljapunov function is strict, i.e. its value strictly decreases along the trajectories of the system, then the only invariant sets, on which the function may be constant, are equilibria.

In the case of a discrete Ljapunov function the image lies in a discrete set and since the function is continuous, the domain of such a function is a union of mutually disjoint sets on which the value of the function is constant. There may be complicated invariant sets inside the sets on which the Ljapunov function is constant.

However, the information implied by the existence of a discrete Ljapunov function is similar to the one implied by the existence of a real valued Ljapunov function; it shows the direction in which the flow flows off the set on which the Ljapunov function is constant.

A *Morse decomposition* $\mathcal{M}(\mathcal{A}) = \{M(p) \mid p \in (\mathcal{P}, \geq)\}$ of an invariant set $\mathcal{A}$ is a decomposition of $\mathcal{A}$ into at most a finite number of disjoint compact invariant subsets $M(p)$, called *Morse sets*, indexed by a partially ordered set $(\mathcal{P}, \geq)$, such that

1. given $x \in \mathcal{A}$ if $\omega(x) \in M(p)$ and $\alpha(x) \in M(q)$ then $q \geq p$
2. if $\omega(x) \in M(p)$ and $\alpha(x) \in M(p)$ then $\varphi(x,t) \in M(p)$ for all t, where $\varphi: \mathcal{A} \times R \to \mathcal{A}$ denotes the flow.

The Morse decomposition exhibits the gradient-like properties of the flow on $\mathcal{A}$ and confines all complicated, reccurent dynamics into individual Morse sets.

Observe, that both the Ljapunov function and the Morse decomposition involve the idea of the flow defined order on the invariant sets. This suggests that there is a strong link between these two concepts. Conley [4], who introduced the notion of a Morse decomposition, showed, that given a Morse decomposition $\mathcal{M}(\mathcal{A}) = \{M(p) \mid p \in (\mathcal{P}, \geq)\}$ of $\mathcal{A}$ there exists a continuous Ljapunov function $V : \mathcal{A} \to \mathbf{R}$ such that

1. if $x \notin \bigcup M(p)$ then $V(x) > V(\varphi(x,t))$ for all $t \geq 0$
2. for each p there exists a number a_p such that $M(p) \subset V^{-1}(a_p)$.

On the other hand given a Ljapunov function $V : \mathcal{A} \to \mathbf{Z}$, one can define a Morse decomposition of $\mathcal{A}$. Let

$$M(p) := \{x \in \mathcal{A} \mid V(\varphi(x,t)) = p \text{ for all t } \}$$

and let the order $(\mathcal{P}, \geq)$ be induced by the order of $\mathbf{Z}$. The collection $\{M(p)\}$ with the order $(\mathcal{P}, \geq)$ is almost a Morse decomposition, but a priori the number of sets in the collection may be infinite. However, one can define a Morse decomposition by choosing an infinite number of appropriate sets $M(p)$ and consider their union together with the union of their mutual connecting orbits as one set in a Morse decomposition. Specific choices depend on the problem considered and we will address this question in the context of $\mathcal{CFS}$, when we use a similar approach in the construction of a Morse decomposition of the global attractor of $\mathcal{CFS}$. As a first step we find a Ljapunov function for the class of $\mathcal{CFS}$ and using this function we define a Morse decomposition of the global attractor $\mathcal{A}$.

It should be noted that this approach is not new. A discrete Ljapunov function was used (under the name of "zero number" or a "lap number") in scalar parabolic equations (for a review article see Fiedler [7]) and in the context of a scalar delay-differential equation with negative feedback it was used to define a Morse decomposition of a global attractor by J. Mallet-Paret [18].

There is an important link between scalar delay-differential equations and cyclic feedback systems. If we discretize a scalar delay equation in time we obtain a cyclic feedback system. More precisely, let us consider a delay equation of the form

$$\dot{x}(t) = f(x(t), x(t-1)).$$

We divide the interval $[-1,0]$ into n equal subintervals and then use the linear approximation of the solution on each subinterval to obtain a cyclic feedback system

$$\begin{aligned}\dot{x}_1 &= f(x_1, x_n)\\ \dot{x}_i &= \frac{1}{n}(x_{i-1} - x_i).\end{aligned}$$

Because of this connection, the Ljapunov function, which we are going to use, is related to the one used for scalar delay equations with negative feedback.

The construction of a discrete Ljapunov function for $\mathcal{CFS}$ is based on the work of Mallet-Paret and Smith [19], who studied monotone cyclic feedback systems.

Definition 1.1 Let $y(t) = \bar{y}(t) - \tilde{y}(t)$ or $y(t) = \dot{y}(t)$ for any two solutions $\bar{y}(t)$, $\tilde{y}(t)$ of a $\mathcal{MCFS}$. Define

$$N(y) = card\{i \mid \delta_i y_i y_{i-1} < 0\}$$

if $y_i \neq 0$ for all i (here we use convention $y_1 = y_{n+1}$). We can extend the domain of definition of N by continuity to

$$\mathcal{N} = \{y \in R^n \mid y_i = 0 \text{ implies } \delta_{i+1}\delta_i y_{i+1} y_{i-1} < 0\}$$

on which N is continuous. If $y \in R^n \setminus \mathcal{N}$ we leave N undefined.

Observe also, that for those $y \in R^n$ with each $y_i \neq 0$, $1 \leq i \leq n$

$$(-1)^{N(y)} = sign \prod_{i=1}^{n} \delta_i y_i y_{i-1} = \prod_{i=1}^{n} \delta_i = \Delta \tag{4}$$

so N takes only odd values if $\Delta = -1$ and only even values if $\Delta = 1$.

A geometrical view of $\mathcal{N}$ may be enlightening. If we denote a open orthant in $\mathbf{R}^n$ by

$$\mathcal{O}(\sigma_1, \ldots, \sigma_n) := \{x \in \mathbf{R}^n \mid \sigma_i x_i > 0\}$$

where $\sigma_i \in \{\pm 1\}$, then we see that $\cup\mathcal{O} \subset \mathcal{N}$ and on each orthant the value of N is constant. Let

$$X_i := \{x \in \mathbf{R}^n \mid x_i = 0,\ \delta_{i+1}\delta_i x_{i+1} x_{i-1} < 0\}$$

denote those parts of coordinate hyperplanes, which are the boundaries of two open orthants on which N has the same value. Then

$$\mathcal{N} = (\cup X_i) \cup (\cup \mathcal{O}).$$

Mallet-Paret and Smith [19] showed that for $\mathcal{MCFS}$ the function N is non-increasing along $y(t)$ i.e. it is a Ljapunov function, see Theorem 2.1. It turns out, that for a general $\mathcal{CFS}$, N is non-increasing only along the difference $y(t) = y(t) - 0$ (note that (2) implies that $y \equiv 0$ is a solution) for any solution $y(t)$. The Ljapunov function can be used to prove other important properties of $\mathcal{MCFS}$. Fusco and Oliva([5]) have shown that for any two hyperbolic periodic orbits o^- and o^+ in $\mathcal{MCFS}$ with $\Delta = 1$ the unstable manifold $W^u(o^-)$ and the stable manifold $W^s(o^+)$ intersect transversally. Their Theorem also can be adapted for the case $\Delta = -1$. Perhaps interestingly, their argument does not apply to the intersection of the stable and unstable manifolds of two hyperbolic fixed points.

1.1 Cyclic feedback systems.

Recall that we assume that (1) has a global compact attractor $\mathcal{A}$. Using the Ljapunov function, described above, we can define a Morse decomposition of $\mathcal{A}$ by

$$M(p) := \{x \in R^n \,|\, \mathcal{N}(\varphi(x,t)) = k \text{ for all } t \in R\}$$

$$\text{where } k = 2p+1 \text{ if } \Delta = -1, \qquad \text{and} \qquad k = 2p \text{ if } \Delta = 1$$

$$M(P) = \{0\} \cup \{\cup_{p \geq P} M(p)\}.$$

The definition of the Morse set $M(P)$ is forced by the fact that the origin 0 is an invariant set, hence it has to be a part of some Morse set. However, the Ljapunov function N is not defined at the origin. The question, where we put the origin in the Morse decomposition is resolved by including it in the highest Morse set $M(P)$. If we do that, the properties of the Morse decomposition require, that if for some $x \in \mathcal{A}$, $\varphi(x,t) \to 0$ as $t \to \infty$, then $x \in M(P)$. Therefore we need to include the set $\cup_{p \geq P} M(p)$ into the Morse set $M(P)$. The value of P, not surprisingly, depends on the dimension of the unstable manifold of the origin for given $\mathcal{CFS}$. It also depends on the sign of the feedback $\Delta = \pm 1$ and on the parity of the dimension n. Let J denotes the number of the eigenvalues with positive real part for the linearization of (1) at the origin. Assume $0 \leq i < n$ and $J > 0$.

$$\text{If } \Delta = -1 \text{ and } n \text{ is odd, then } \quad P = \begin{cases} \frac{n+1}{2} & \text{if } J = n \\ i-1 & \text{if J=2i,2i+1.} \end{cases}$$

$$\text{If } \Delta = 1 \text{ and } n \text{ is odd, then } \quad P = \begin{cases} \frac{n+1}{2} & \text{if } J = n \\ i-1 & \text{if J=2i-1,2i.} \end{cases}$$

$$\text{If } \Delta = -1 \text{ and } n \text{ is even, then } \quad P = \begin{cases} \frac{n}{2} & \text{if } J = n \\ i-1 & \text{if J=2i,2i+1.} \end{cases}$$

$$\text{If } \Delta = 1 \text{ and } n \text{ is even, then } \quad P = \begin{cases} \frac{n+2}{2} & \text{if } J = n \\ i-1 & \text{if J=2i-1,2i.} \end{cases}$$

For a proof that the above collection is a Morse decomposition and further details see Gedeon and Mischaikow [6]. Observe, that $M(p)$ is the invariant set which lies in the union of orthants, on which the function N assumes the value k.

The concept of a Morse decomposition provides us with an important framework in which we will formulate our questions. In particular, it gives the following meaning to the words global and local. Global dynamics in this framework is the gradient-like structure of the flow outside of the individual Morse sets. When one studies the global dynamics in this sense, the object of interest are the connecting orbits between different Morse sets and their structure. Local dynamics in this framework is the dynamics within the individual Morse sets. Since this may involve rich dynamics, including chaos, it is by no means a local dynamics in the usual sense i.e. dynamics controlled by the linear part of the vector field.

We use these concepts to ask two natural questions:

Q1 What is the global dynamics of $\mathcal{CFS}$?

Q2 What may be the possible local dynamics of $\mathcal{CFS}$?

It is obvious, that the local dynamics of $\mathcal{CFS}$ will depend on the system under consideration. The beauty of the idea of a Morse decomposition lies in the fact that the first question can be answered in its generality. Thus our quest to describe the dynamics on the global attractor has two parts; description of the Morse decomposition and the structure of the connecting orbits between the Morse sets, which is the part common to all systems (1) and the description, in the form of examples, of how complicated the individual Morse sets may be.

Before we attempt to answer the first question, we will discuss the existence of periodic orbits for $\mathcal{CFS}$, since this is one of the questions relevant to biological models using $\mathcal{CFS}$.

Recall that the *maximal invariant set* in B is defined by

$$Inv\ B := \{x \in \mathbf{R}^n \mid \varphi(x,t) \in B \text{ for all } t\}.$$

The set B is called an *isolating neighborhood* if it is closed and $Inv\ B \subset \text{int}(B)$. The set $\Xi \subset \mathbf{R}^n$ is a *local section* for φ if there exists a time $T > 0$ such that $\varphi((-T,T),\Xi)$ is an open subset of $\mathbf{R}^n$ homeomorphic to $(-T,T) \times \Xi$. Ξ is a *Poincaré section* for a Morse set $M(p)$ if

1. it is a local section,
2. given an isolating neighborhood N_p of $M(p)$, $\Xi \cap N_p$ is closed,
3. for every $x \in M(p)$, $\varphi((0,\infty),x) \cap \Xi \neq \emptyset$.

Since the flow flows in one direction from the union of orthants, on which $N(\cdot) = k_1$, to the union of orthants, on which $N(\cdot) = k_2$ with $k_1 > k_2$, isolating neighborhood N_p of $M(p)$ can be chosen to be a closed subset of the union of orthants in which $M(p)$ lies. Since, on X_i, $\dot{x}_i \neq 0$, the open subsets of the X_i are local sections for the flow and ideal candidates for Poincaré sections.

We will call a periodic orbit $x(t)$ *large* if, for all $i = 1,\ldots,n$, there are times t_i and t_i' such that $[x(t_i)]_i > 0, [x(t_i')]_i < 0$, where $[x]_i$ denotes the i-th coordinate of x. Otherwise a periodic orbit will be called *small.* Large periodic orbits describe oscillations with the property, that each variable changes sign twice during one period of oscillation.

We will denote by $\mathcal{CFS}^-$ a cyclic feedback system with $\Delta = -1$ and by $\mathcal{CFS}^+$ with $\Delta = +1$.

Theorem 1.2 (Gedeon and Mischaikow, [6]) *For $\mathcal{CFS}^+$ let $p = 1,\ldots,P-1$ and for $\mathcal{CFS}^-$ let $p = 0,\ldots,P-1$. Let $J < n$.*

1. *There exists an essential continuous surjective map*

$$\theta_p : M(p) \to S^1$$

   where S^1 is the unit circle.

2. *If, for some $i = 1,\ldots,n$, the set X_i is a Poincaré section of $M(p)$, then $M(p)$ contains a large periodic orbit.*

3. *If, in addition, one considers a $\mathcal{MCFS}$ and if $M(p)$ contains no fixed points then $M(p)$ contains a large periodic orbit.*

We want to remark, that the assumption of no fixed points in $M(p)$ implies the existence of a Poincarè section for $\mathcal{MCFS}$. It is an open problem whether the same implication holds for a general $\mathcal{CFS}$.

Theorem 1.2 generalizes considerably the result of Hastings et.al. [9] who showed that for $\mathcal{MCFS}$ with $\Delta = -1$ (and under some additional conditions), if the origin is unstable, then there exists a non-constant periodic solution. Their approach utilized the Brouwer fixed point theorem.

1.2 Global dynamics of $\mathcal{CFS}$.

Now we attempt to answer the question **Q1**. For the proofs of the results, the reader should consult the original paper of Gedeon and Mischaikow [6].

Recall that we assume that $\mathcal{CFS}$ admits a global attractor $\mathcal{A}$. To characterize the dynamics on this attractor we construct a model flow and then relate it to the flow on the attractor $\mathcal{A}$.

We define the model flow as follows. Let A be a $k \times k$ matrix of the form

$$A = \begin{bmatrix} A_0 & 0 & \dots & 0 \\ 0 & A_1 & & \\ & & \ddots & \\ 0 & & & A_{P-1} \end{bmatrix}.$$

The submatrices A_p, $p = 0, \dots, P-1$ have two forms:

$$A_p = \frac{1}{p+1} \qquad \text{(Type I)}$$

and

$$A_p = \begin{bmatrix} (p+1)^{-1} & 2\pi \\ -2\pi & (p+1)^{-1} \end{bmatrix}. \qquad \text{(Type II)}$$

Let $z = (z_0, \dots, z_{k-1}) \in \mathbf{R}^k$. Then in polar coordinates $z = r\zeta$ where $r \geq 0$ and $\zeta \in S^{k-1}$, the unit sphere in $\mathbf{R}^k$. Let $D^k = \{z = (z_0, \dots, z_{k-1}) \mid \sum_{p=0}^{k-1} z_p^2 \leq 1\}$ be the closed unit ball in $\mathbf{R}^k$. Consider the flow

$$\psi : \mathbf{R} \times D^k \to D^k \tag{5}$$

generated by the equations

$$\dot{\zeta} = A\zeta - \langle A\zeta, \zeta \rangle \zeta \tag{6}$$

$$\dot{r} = r(1-r). \tag{7}$$

The dynamics of ψ is most easily understood if one observes that (6) is obtained by projecting the linear system $\dot{z} = Az$ onto the unit sphere.

The value of k and the choice of Type I or Type II matrices is determined by the $\mathcal{CFS}$. In particular k is related to the number of eigenvalues with positive real part of the matrix obtained by linearizing about $\mathbf{0}$ in the $\mathcal{CFS}$. We shall denote the cyclic feedback system by $\mathcal{CFS}_{even}$ and $\mathcal{CFS}_{odd}$ if n is even or odd, respectively.

The specific choices for the A_p's as a function of the type of $\mathcal{CFS}$ are as follows:

$\mathcal{CFS}^-_{odd}$: A_p, $p = 0, \ldots P-1$ are of Type II unless $n = 2P+1$ when A_p, $p = 0, \ldots, P-2$ are of Type II and A_{P-1} is of Type I.

$\mathcal{CFS}^+_{odd}$: A_0 is of Type I and A_p, $p = 1, \ldots P-1$ are of Type II.

$\mathcal{CFS}^-_{even}$: A_p, $p = 0, \ldots P-1$ are of Type II.

$\mathcal{CFS}^+_{even}$: A_0 is of Type I and A_p, $p = 1, \ldots, P-1$ are of Type II unless $n = 2P$ when A_p, $p = 1, \ldots, P-2$ are of the Type II and A_{P-1} is of Type I.

When it is necessary to distinguish between the model flows we shall let $\psi^\pm_*$ denote the corresponding flow where $*$ denotes *even* or *odd*.

Let $\Pi(p)$, $p = 0, \ldots, P-1$ denote the invariant set of ψ in the invariant subspace corresponding to A_p and let $\Pi(P) := \mathbf{0}$, the origin. Observe that $\{\Pi(p) \mid p = 0, \ldots, P\}$ forms a Morse decomposition of ψ on D^k.

Theorem 1.3 (Gedeon and Mischaikow, [6]) *Consider $\mathcal{CFS}^\pm_*$. Assume that, if A_p is of Type II, then $M(p)$ has a Poincaré section. Then there exist a continuous surjective function*

$$\rho : \mathcal{A} \to D^k$$

for which $M_p = \rho^{-1}(\Pi(p))$ $(p = 0, \ldots, P)$ and a continuous flow $\tilde{\varphi} : \mathbf{R} \times \mathcal{A} \to \mathcal{A}$ obtained via an order preserving time reparameterization of φ such that the following diagram commutes

$$\begin{array}{ccc} \mathbf{R} \times \mathcal{A} & \xrightarrow{id \times \rho} & \mathbf{R} \times D^k \\ \downarrow \tilde{\varphi} & & \downarrow \psi^\pm_* \\ \mathcal{A} & \xrightarrow{\rho} & D^k \end{array}$$

i.e. φ is semi–conjugate to $\psi^\pm_*$.

An immediate corollary of Theorem 1.2 and Theorem 1.3 is as follows.

Corollary 1.4 *Consider $\mathcal{MCFS}^\pm_*$ and assume that if A_p is of Type II, then $M(p)$ has no fixed points. Then there exists a semi–conjugacy from $\tilde{\varphi}$ to $\psi^\pm_*$.*

We want to take some time to explain the implications of Theorem 1.3 to the dynamics of $\mathcal{CFS}$.

One conclusion follows immediately from Theorem 1.2.2; if $M(p)$ is mapped by the semi–conjugacy ρ to a periodic orbit, then $M(p)$ contains a large periodic orbit.

However, a set $M(p)$ may be more complicated then a periodic orbit. The question how complicated it may be is precisely the question **Q2** and will be addressed later in this paper.

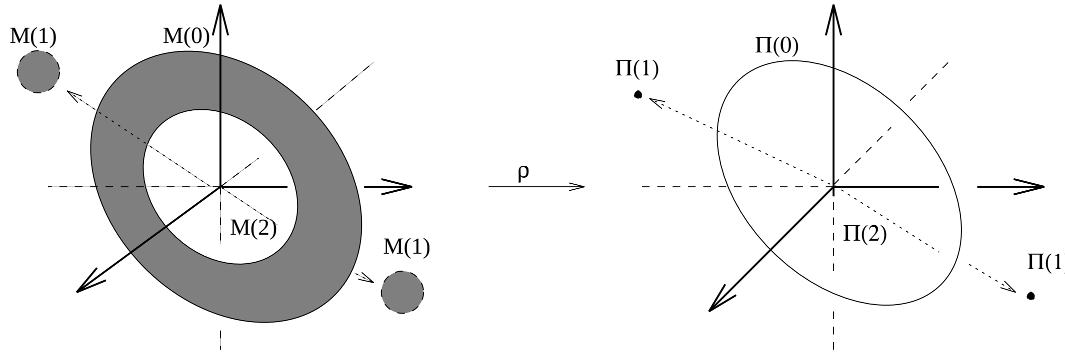


Figure 1: Semi–conjugacy in the case $n = 3$ with feedbacks $\delta_1 = -1$, $\delta_2 = \delta_3 = 1$.

Let us just say for now that Theorem 1.11 and Theorem 1.13 below imply that there are cyclic feedback systems, where the set $M(0)$ contains at least a suspension of a shift dynamics on two symbols. As we see in this case the semi–conjugacy ρ maps a complicated invariant set $M(0)$ onto a periodic orbit in the model flow (Figure 1.2).

In this particular example we lost a tremendous amount of information via the semi–conjugacy. On the other hand, this deliberate negligence allowed us to characterize the dynamics on the global attractor for the whole class of $\mathcal{CFS}$.

The other implication of Theorem 1.3 and Corollary 1.4 concerns the existence of connecting orbits between Morse sets. For the model flow ψ the stable and unstable manifolds for the Morse sets $\Pi(p)$ are known explicitly. Hence the set of connecting orbits between any two Morse sets, $C(\Pi(p), \Pi(q))$, is also known explicitly.

Returning to the cyclic feedback system we now conclude that

$$C(M(p), M(q)) = \rho^{-1}(C(\Pi(p), \Pi(q))).$$

In analogy with the discussion above, how well $C(\Pi(p), \Pi(q))$ "approximates" the set of connecting orbits $C(M(p), M(q))$ depends on how much dynamics is being lost through the semi–conjugacy. In particular, if we return to the example given by Theorem 1.11 and Theorem 1.13, then the set of connecting orbits $C(M(1,2), M(0))$, (where $M(1,2) := M(2) \cup M(1) \cup C(M(2), M(1)))$, through the preimage of $S^1 \times (0,1) \times (0,1)$, must be fairly complicated since the suspension of the shift on two symbols is contained in the closure of $C(M(1,2), M(0))$.

Obviously, the hypotheses of Theorem 1.3 will not always be satisfied; there may occur fixed points in $M(p)$ and even if there are no fixed points we conjecture that there are $\mathcal{CFS}$ for which $M(p)$ will not have a Poincaré section. In these cases we have no hope of controlling the dynamics in the Morse sets and hence there is no hope of constructing a semi–conjugacy. There are two ways how to address this problem. We can restrict the class of systems and show that in the restricted class open subsets of X_i act as Poincarè sections for appropriate Morse sets. We can also introduce a weaker notion of comparison between flows on invariant sets.

Let us follow both ideas.

Theorem 1.5 (Gedeon and Mischaikow, [6]) *(Sufficient condition for the existence of*

Poincarè sections.)

Consider $\mathcal{CFS}$ of the form

$$\dot{x}_i = \alpha_i g_i(x_i) + \beta_i f_i(x_{i-1}), \qquad i = 1, \ldots, n$$

$\alpha_i, \beta_i \in \{\pm 1\}$ *and we assume that for every* i $x_i g_i(x_i) > 0$ *and* $x_{i-1} f_i(x_{i-1}) > 0$.

If $\prod_{i=1}^{n} \alpha_i \beta_i = (-1)^{n+1}$ *then for every* i, X_i *is a Poincarè section.*

Now we follow the second idea. Let us recall that given a Morse decomposition $\mathcal{M}(\mathcal{A}) = \{M(p) \mid p \in (\mathcal{P}, >)\}$ an *interval* $I \subset \mathcal{P}$ satisfies the property that if $p, q \in I$ and $p > r > q$, then $r \in I$. The importance of intervals is that given a Morse decomposition all coarser Morse decompositions involve isolated invariant sets of the form

$$M(I) := \left(\bigcup_{p \in I} M(p) \right) \bigcup \left(\bigcup_{p,q \in I} C(p,q) \right)$$

where I is an interval.

Definition 1.6 If $\mathcal{A}$ and $\mathcal{B}$ are invariant sets with Morse decompositions $\mathcal{M}(\mathcal{A}) = \{M(p) \mid p \in (\mathcal{P}, >)\}$ and $\mathcal{M}(\mathcal{B}) = \{M(q) \mid q \in (\mathcal{Q}, >)\}$ respectively, then the Morse decomposition $\mathcal{M}(\mathcal{A})$ is *topologically semi-equivalent* to the Morse decomposition $\mathcal{M}(\mathcal{B})$ if there exists

1. an order preserving bijection $\bar{\rho} : \mathcal{P} \to \mathcal{Q}$, and

2. a continuous surjection $\rho : \mathcal{A} \to \mathcal{B}$ such that

$$M(I) = \rho^{-1}(M(\bar{\rho}(I)))$$

   for every interval $I \subset \mathcal{P}$.

Theorem 1.7 (Gedeon and Mischaikow, [6]) *Given a cyclic feedback system* $\mathcal{CFS}_*^{\pm}$*, the Morse decomposition* $\mathcal{M}(\mathcal{A})$ *is topologically semi-equivalent to* $\mathcal{M}(D^K, \psi_*^{\pm})$.

In this description of the global attractor $\mathcal{A}$ the dynamics is almost completely ignored. The map ρ does not give any information about particular orbits and exhibits only a topological structure of invariant sets in $\mathcal{A}$.

However, we observe that stable manifolds are mapped onto stable manifolds and the unstable manifolds are mapped onto unstable manifolds. As an example of the information this description provides, let us assume that $\Delta = -1$, $n \geq 4$ and $P \geq 2$. Then the set $M(0,1) := M(0) \cup M(1) \cup C(M(1), M(0))$ is a preimage of an essential map onto $\Pi(0,1) := \Pi(0) \cup \Pi(1) \cup C(\Pi(1), \Pi(0))$. One can easily check that $\Pi(0,1)$ is homeomorphic to S^3 a 3-dimensional sphere.

1.3 Local dynamics of $\mathcal{CFS}$.

Now we consider the question **Q2**, which in the view of the results above can be restated as follows: How much information do we loose by the semi–conjugacy ρ?

For a subclass $\mathcal{MCFS}$ there is a general answer to the question **Q2**, which takes the form of a Poincarè-Bendixon type result.

Theorem 1.8 (Mallet-Paret and Smith, [19]) *Let us consider a $\mathcal{MCFS}$ in R^n. Consider any point x and its omega limit set $\omega(x)$. Then $\omega(x)$ is of one of the of the following:*
i) a fixed point
ii) a limit cycle
iii) a set $H = E \cup C$ where E is set of equilibria and C is the set of connecting orbits between the equilibria in E.

The aim of this paper is to answer question **Q2** in the framework of general $\mathcal{CFS}$. Since the local dynamics depends on the system chosen and because of Theorem 1.8, the answer will take the form of an example of a particular subclass of $\mathcal{CFS}$, disjoint with $\mathcal{MCFS}$, which exhibits chaotic behavior. It is worth noting, that the complicated invariant set, though locally unstable, is a subset of the lowest Morse set $M(0)$, which is the most attracting Morse set in the framework of a Morse decomposition.

A usual way to exhibit the presence of a complicated dynamics is to relate the dynamics in the system under consideration to a *shift dynamics*. In this paper we will use the full shift on two symbols.

Let $\Sigma := \Pi_{n=-\infty}^{\infty}\{0,1\}_n$ be a space of doubly infinite sequences consisting of 0's and 1's. We define a metric on Σ by

$$d(\alpha,\beta) = \sum_{i=-\infty}^{\infty} \frac{|\alpha_i - \beta_i|}{2^{|i|}}$$

where $\alpha, \beta \in \Sigma$ and α_i denotes the i-th component of α.

A shift map $\sigma : \Sigma \to \Sigma$ is defined by

$$\sigma(\alpha)_{i-1} = \alpha_i.$$

The map σ is a prototype of a chaotic map since it has several properties, which are usually attributed to such maps, for instance

- periodic orbits are dense in Σ
- there exists an $x \in \Sigma$ whose trajectory is dense in Σ
- for any $\epsilon > 0$ and any two non periodic points x and y, such that $d(x,y) < \epsilon$ there is an n such that $d(\sigma^n(x), \sigma^n(y)) \geq 1$.

We will start the construction of an $\mathcal{CFS}$ with a chaotic dynamics by first looking at the following $\mathcal{MCFS}$ with negative ($\Delta = -1$) feedback:

$$\begin{aligned} \dot{x}_1 &= -a_1 x_1 - b_1 f(x_n) \\ \dot{x}_i &= -a_i x_i + b_i x_{i-1} \qquad i = 2, \ldots, n \end{aligned} \tag{8}$$

where we assume that f is a monotone C^1 function with

$$xf(x) > 0 \qquad \text{if } x \neq 0.$$

Without loss of generality we assume $f'(0) = 1$.

The usual way one associates the dynamics of a shift map to the dynamics generated by the system of ordinary differential equations is to consider a periodic orbit γ of this system and the associated Poincarè map Π. Then Π has a fixed point p corresponding to the periodic orbit γ. If one establishes that the stable and unstable manifolds of p under Π intersect transversally, then the classical result of Smale [23] and Moser [22] asserts, that there is a conjugacy between a Poincarè map on some invariant set S and a shift map σ on Σ. This means, that there is a homeomorphism $\bar{\rho}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbf{S} & \xrightarrow{\ \Pi\ } & \mathbf{S} \\ {\scriptstyle \bar{\rho}}\downarrow & & \downarrow{\scriptstyle \bar{\rho}} \\ \Sigma & \xrightarrow{\ \sigma\ } & \Sigma \end{array}$$

We will partially follow the same strategy. Let us assume for the moment that the system (8) admits a hyperbolic periodic orbit γ with the one dimensional unstable manifold. In this paper, the dimension of the unstable manifold of the periodic orbit is the number of Floquet multipliers with absolute value bigger than one.

We construct a nonlinearity $\hat{g}$ such that the system

$$\begin{aligned} \dot{x}_1 &= -a_1 x_1 - b_1 \hat{g}(x_n) \\ \dot{x}_i &= -a_i x_i + b_i x_{i-1} \qquad i = 2, \ldots, n \end{aligned} \tag{9}$$

which we obtain from (8) by replacing f by $\hat{g}$ admits a homoclinic orbit to γ. This implies that there will be a homoclinic orbit to p under the corresponding Poincarè map Π, which means that the stable and unstable manifolds of p under Π will have a nonempty intersection.

This far we follow what we call a classical strategy, which was outlined above.

However, we are not able to show that the intersection of the stable and unstable manifolds of p under Π is transversal and so we cannot assert the existence of the conjugacy between Π and σ.

What we show instead is that these manifolds have a *topological crossing.* We postpone the definition of a topological crossing to Section 2, but intuitively this means that the manifolds have a "robust" crossing, which cannot disappear under small perturbations, though it does not have to be a transversal crossing. Then we use ideas, developed independently by Burns and Weiss [2] and Gedeon, McCord and Mischaikow [8] to tie the existence of a topological crossing to the existence of a semi-conjugacy of Π and σ.
It should be noted that a similar strategy was used by B. Lani-Wajda and H-O. Walther [16, 17] to prove that a delay-differential equation with negative feedback may exhibit a complicated behavior. They constructed a function $g(x)$ with $xg(x) > 0$ for $x \neq 0$, such that

$$\dot{x} = -g(x(t-1))$$

has a periodic solution with the property that the stable and unstable manifolds of the corresponding fixed point of a Poincarè map intersect transversally. Thus in their case there is a conjugacy between a Poincarè map on some invariant set S and a shift map σ on Σ.

Let us now search for a possible class of nonlinearities $\hat{g}$.

Theorem 1.9 *Let G be a nontrivial periodic orbit in a $\mathcal{MCFS}$. Let W^s and W^u be the stable manifold and unstable manifolds of G respectively.*
Then $(W^s \setminus G) \cap (W^u \setminus G) = \emptyset$.

Observe, that this Theorem shows that we have to search for $\hat{g}$ outside of the class of $\mathcal{MCFS}$, as should be expected because of Theorem 1.8. The proof of Theorem 1.9 will be given in Section 2, using the Ljapunov function N in the spirit of the work of Mallet-Paret and Smith [19].

Further limitation on the choice of $\hat{g}$ are placed by the requirement, that system (9) should be a $\mathcal{CFS}$ and by the fact that, in the outlined strategy, we implicitly assumed that the periodic orbit γ is a periodic orbit of the new system (9).

Let $\Phi(x,t)$ be the flow generated by (8) and let $[x]_j$ denotes the j-th coordinate of a point x. If $\gamma(t)$ is a periodic solution of system (8) then we will denote $\gamma = \{\gamma(t)\}$ the orbit defined by the solution $\gamma(t)$.

Definition 1.10 Given a periodic orbit γ, let $M = (M_1, \ldots, M_n) \in \gamma$ such that $M_n = \max_{0 \leq t \leq \bar{T}} [\gamma(t)]_n$, where $\bar{T}$ is the minimal period of $\gamma(t)$, be a point with maximal n-th coordinate value on γ.

We will consider nonlinearities of the following form (Figure 2):

$$\begin{aligned} \hat{g}(x) &= f(x) \text{ if } x \in (-\infty, M_n + \delta] \\ \frac{d}{dx}(\hat{g}(x)) &< 0,\ 0 < \hat{g}(x) \leq L \text{ if } x \in (M_n + \delta + \eta, \infty) \\ \hat{g}(x) & \quad \text{has a unique maximum in } y \in (M_n + \delta, M_n + \delta + \eta] \\ & \quad \text{with } \hat{g}(y) < f(y) \end{aligned} \tag{10}$$

There are three constants δ, η, L in this definition. We will assume that $0 < L < f(M_n)$ so that the second and third line of the definition makes sense and we take $\delta, \eta > 0$.

Observe, that by the requirement $\hat{g}(x) > 0$ the system (9) is a $\mathcal{CFS}$ for all $\hat{g}$ of the form (10). Since $f = \hat{g}$ in the range of the periodic orbit γ, γ will be a periodic orbit for (9) for any $\hat{g}$ of the form (10).

Now we fix some notation. Recall, that we assume that (8) admits a periodic orbit γ with one-dimensional unstable manifold. Let (TW, π) be the tangent vector bundle to $\mathbf{R}^n$, restricted to γ, equipped with the linear vector field

$$\dot{y} = A(t)y$$

where $A(t)$ is the linearization of (8) along $\gamma(t)$. Since γ is hyperbolic, (TW, π) admits a splitting

$$(TW, \pi) = (TW^u, \pi_u) \oplus \Gamma \oplus (TW^s, \pi_s)$$

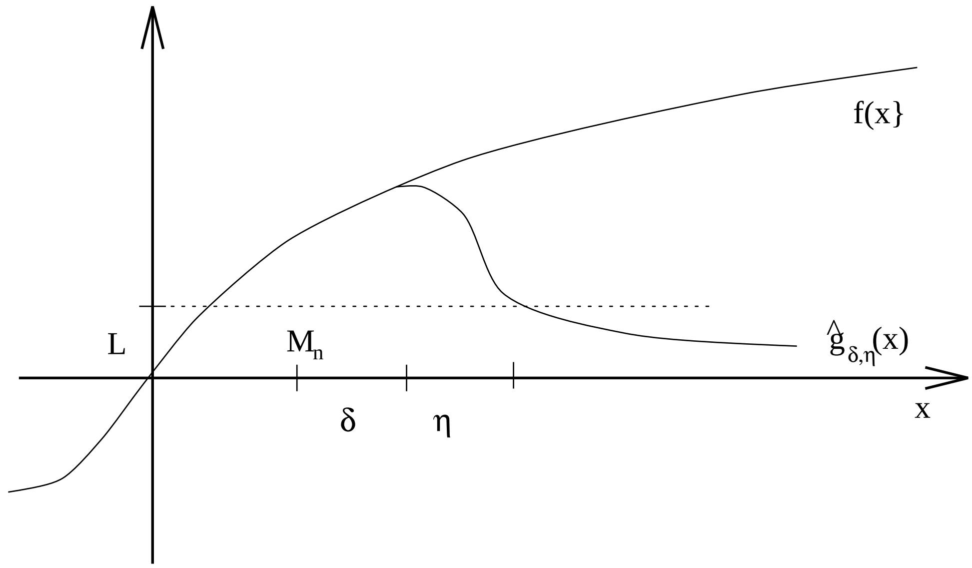


Figure 2: Functions $f(x)$ and $\hat{g}_{\delta,\eta}(x)$.

with $\Gamma(t) = \{v \in TW \mid v = c\dot{\gamma}(t)\}$ and the estimates with $\beta_s, \beta_u > 0$;

$$||y \cdot t|| \leq ce^{-\beta_s t}||y|| \text{ for } y \in TW^s \text{ and } t \geq 0 \tag{11}$$

$$||y \cdot t|| \geq ce^{\beta_u t}||y|| \text{ for } y \in TW^u \text{ and } t \geq 0. \tag{12}$$

Let $TW^u_\Gamma(t) := span\ \{TW^u(t), \Gamma(t)\}$ and $TW^s_\Gamma(t) := span\ \{TW^s(t), \Gamma(t)\}$ be the fibers of subbundles TW^u_Γ and TW^s_Γ respectively. By the assumption $dim\, TW^s_\Gamma(t) = n-1$ and $dim\, TW^u_\Gamma = 2$.

Let $\rho^u := e^{\beta_u \bar{T}}$, where $\bar{T}$ is a period of $\gamma(t)$, be the Floquet multiplier bigger then 1 and let $\rho^s := e^{\beta_s \bar{T}}$ be the inverse of the multiplier with largest absolute value less then 1.

We assume for the rest of the paper that the time along $\gamma(t)$ is reparametrized in such a way that $M = \gamma(0)$.

Let us denote the unit normal vector to $TW^s_\Gamma(0)$ by α and let

$$H := \{x \in \mathbf{R}^n \mid [\dot{x}]_n = 0\} = \{x \in \mathbf{R}^n \mid [x]_n = \frac{a_n}{b_n}[x]_{n-1}\}.$$

We will show later that $H \cap TW^u_\Gamma(0) \neq \emptyset$ and H is transversal to the flow at $M \in \gamma$. Hence $H \cap TW^u_\Gamma(0)$ is one-dimensional and we let $m = (m_1, \ldots, m_n)$ to be a unit vector in $H \cap TW^u_\Gamma(0)$.

Our results are based on the following Theorem. It says not only that there exists a function $\hat{g}$ for which the system (8) with $\hat{g}$ instead of f admits a chaotic dynamics, but also characterize the set of parameters L, δ, η for which such a function exists.

Theorem 1.11 *Assume that the system (8) in $\mathbf{R}^n$ admits a hyperbolic periodic orbit γ with one-dimensional unstable manifold. Fix L such that $0 < L < f(M_n)$ and choose a two dimensional family $\mathcal{G} : (\delta, \eta) \to \hat{g}_{\delta,\eta}$ of functions of the form (10).*

Fix any one dimensional family $\mathcal{F} \subset \mathcal{G}$ parameterized by a continuous curve of the form $(\delta, \eta(\delta))$ in the neighborhood of $(0,0)$ such that

$$\eta(\delta) = O(\delta^q), \qquad q > 2 \qquad \text{as } \delta \to 0. \tag{13}$$

If either

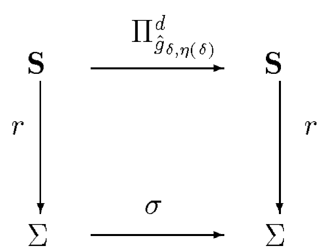
a.

$$n = 3 \qquad\qquad or$$

b.

$$(C+2)\rho^u < \rho^s \tag{14}$$

where $C = \sqrt{(1 - \frac{m \cdot \alpha}{[\alpha]_1})^2 + \sum_{i=2}^n m_i^2}$, then there exist $\bar{\delta} = \bar{\delta}(\mathcal{F}, \mathcal{G}) > 0$ such that for any $\delta \leq \bar{\delta}$ the system (9) with $\hat{g}_{\delta,\eta(\delta)} \in \mathcal{F}$ admits $N \subset H$ as a Poincarè section with a Poincarè map $\Pi_{\hat{g}_{\delta,\eta(\delta)}}$ and there exist an invariant set $S \subset N$, a continuous surjective map $r : S \to \Sigma$ and an integer d such that the following diagram commutes



i.e. $(S, \Pi^d_{\hat{g}_{\delta,\eta(\delta)}})$ is semi–conjugate to (Σ, σ).

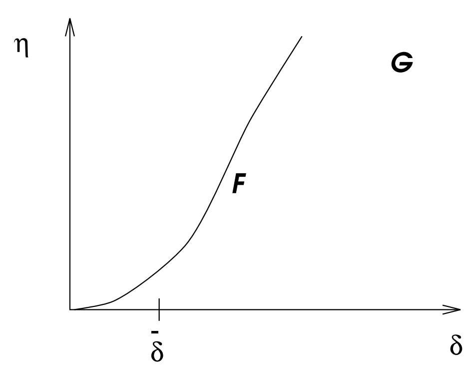


Figure 3: A possible set of pairs (δ, η) which parameterize family $\mathcal{F}$.

Observe, that since the condition (13) is open the set of (δ, η) for which the Theorem holds is open in $\mathbf{R}^2$. Note, that the result does assert the existence of a semi–conjugacy and not a conjugacy which means that the map r is not necessarily one-to-one. We would also like to remark that in condition (14) all involved quantities are determined by the linear variational system around γ. In particular, the constant C depends only on the mutual position of $TW^s_\Gamma(0)$ and $TW^u_\Gamma(0)$. We will also show that $[\alpha]_1 \neq 0$ and so C is always well defined.

Let us remark, that the system (9) with $n = 3$ and $\hat{g} := \hat{g}_{\delta,\eta(\delta)}(x) \in \mathcal{F}$ for $\delta \leq \bar{\delta}$ satisfying the Theorem 1.11 is the simplest possible system which may have a chaotic behavior, since the phase space is 3-dimensional and there is only one nonlinear term on the right-hand side.

The result was obtained by altering a $\mathcal{MCFS}$ into a $\mathcal{CFS}$ by changing the function $f(x)$ into a function $\hat{g}_{\delta,\eta(\delta)}(x)$. However, the functions f and $\hat{g}_{\delta,\eta(\delta)}(x)$ are not close in any function space. A natural question is, whether we can achieve the same result by a small perturbation of the function f. The answer is positive.

Theorem 1.12 *Assume that the system (8) in* $\mathbf{R}^n$ *admits a hyperbolic periodic orbit* γ *with one-dimensional unstable manifold.*

If either

- $n = 3$ *or*
- $(C + 2)\rho^u < \rho^s$

then there is $N \subset H$ *and for every* ϵ *there is a function* $h \in C^1(\mathbf{R}, \mathbf{R})$ *with*

$$||f - h||_{C^0} < \epsilon$$

with the following properties.

The system (8) with f *replaced by* h *admits* N *as a Poincarè section with a Poincarè map* π *and there exist an invariant set* $S \subset N$*, a continuous surjective map* $r : S \to \Sigma$ *and an integer* d *such that the following diagram commutes*

$$\begin{array}{ccc} \mathbf{S} & \xrightarrow{\pi^d} & \mathbf{S} \\ r\downarrow & & \downarrow r \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array}$$

i.e. (S, π^d) *is semi–conjugate to* (Σ, σ).

This result is interesting because, as we saw in Theorem 1.8, for $\mathcal{MCFS}$ a Poincarè-Bendixon trichotomy holds and by the result of Tereščák [25] a C^1 perturbation of a $\mathcal{MCFS}$ will preserve the Poincaré-Bendixon properties of the flow.

In more limited setting the C^1-perturbation result is due to M. Hirsch. Observe that if n is odd and we change $t \to -t$ in the flow generated by (8) then all the feedbacks δ_i change the sign and we get a $\mathcal{MCFS}$ with $\Delta = 1$. Such a flow defines a monotone dynamical system. For $n = 3$ such a system cannot exhibit chaotic dynamics; furthermore, this property is stable under C^1 perturbations of the flow (Hirsch [12],[13]). Again Theorem 1.12 provides a concrete example of the fact that this property is not stable under C^0 perturbation.

Undoubtly every reader now wonders, whether the assumptions of Theorem 1.11 and Theorem 1.12 can be verified in the class of systems (8). In particular, we need to show that there are systems in this class which admits a hyperbolic periodic orbit γ with one-dimensional unstable manifold. Theorem 1.13 below shows that this is the case for any n.

The condition (14) is more difficult to verify, since the constant C is hard to compute in the concrete examples. Since it depends only on the mutual position of $TW^s_\Gamma(0)$ and $TW^u_\Gamma(0)$ at the point $M = \gamma(0)$ one is tempted to construct an example of a system which satisfies (14) in a following way. Let us consider a one-dimensional family of systems (8) possessing a hyperbolic periodic orbit γ, where the eigenvectors of the Poincarè maps, corresponding to γ, are fixed throughout the family but the ratio $\frac{\rho^s}{\rho^u}$ increases without bounds. Then one would argue that whatever C is, for some some system from the family, condition (14) will be satisfied. Unfortunately, it is not clear, if this construction can be carried out in the class of systems (8). The special structure of the equations forces special structure of the eigenvalues and eigenvectors (see section 5) and it may be the case that one cannot change Floquet multipliers without changing eigenvectors of the Poincaré map.

The construction of the systems which verify (14) for $n \geq 4$ remains an open problem.

Theorem 1.13 *Let f be a monotone C^3 function with $f'(0) = 1$ and $xf(x) > 0$ if $x \neq 0$. Furthermore, we assume $f''(0) = 0$, and that $f'''(0) =: d > 0$. We fix n and all the coefficients a_i, $i = 1, \ldots, n$, b_j, $j = 2, \ldots, n$ in the system (8).*

If we treat b_1 as a bifurcation parameter, there is a $b_1^* = b_1(a_i, b_j, n)$, $i = 1, \ldots, n$, $j = 2, \ldots, n$, when the origin undergoes a subcritical Hopf bifurcation and for sufficiently small $\alpha := b_1 - b_1^* < 0$ there is a hyperbolic periodic orbit with one-dimensional unstable manifold.

The paper is organized as follows.

In Section 2 we review some results for $\mathcal{MCFS}$ and prove Theorem 1.9. We also prove some useful facts about the stable and unstable manifolds of γ. Finally, we define the topological crossing of manifolds, state a theorem which ties the topological intersection to the shift dynamics and show, how one can verify that two manifolds have a topological crossing.

In Section 3 we prove the main results Theorem 1.11 and Theorem 1.12. Since the proof of Theorem 1.11 is rather technical, we decided to present the proof in three levels. We first discuss the main ideas of the proof, which are illustrated by pictures in a 3-dimensional setting. The section itself then presents all the ideas in detail; however proofs of the most technical lemmas are delegated to Section 4.

The proof of Theorem 1.13 occupies Section 5.

2 Linear theory.

2.1 Monotone cyclic feedback systems.

The main aim of this section is the proof of Theorem 1.9.

We start to develop our main tool in the study of $\mathcal{MCFS}$ – a discrete Ljapunov function, which was defined in the introduction. The name Ljapunov function is justified by the following result.

Theorem 2.1 (Mallet-Paret and Smith,[19]) *Let $y(t) = \bar{y}(t) - \tilde{y}(t)$ or $y(t) = \dot{\bar{y}}(t)$ for any two solutions $\bar{y}(t)$, $\tilde{y}(t)$ of a $\mathcal{MCFS}$. Then*

1. *$y(t) \in \mathcal{N}$ except for isolated values of t;*
2. *$N(y(t))$ is locally constant for $y(t) \in \mathcal{N}$;*
3. *if $y(t_0) \notin \mathcal{N}$ then $N(y(t_0^+)) < N(y(t_0^-))$ where $t_0^+ > t_0 > t_0^-$;*
4. *if $y(t) \in \mathcal{N}$ then $(y_i(t), y_{i-1}(t)) \neq (0,0)$ for all i;*
5. *if $y(t) \in \mathcal{N}$ then $(y_i, \dot{y}_i) \neq (0,0)$ for all i.*

Let us consider

$$\dot{x} = A(t)x, \quad A(t+T) = A(t) \tag{15}$$

the $n-$dimensional linear system and assume it is a monotone cyclic feedback system. This means, that if we denote $A(t) = [a_{ij}(t)]$ then $a_{ij}(t) \equiv 0$ for (i,j) with $j \neq i, i-1$ and $a_{i,i-1}(t)$ has a constant sign for $0 \leq t \leq T$. This implies that the property (3) holds for (15).

Let $X(t)$ be the fundamental matrix solution with $X(0) = I$. Define for a given $\alpha \in C \setminus \{0\}$ the complex eigenspaces

$$E_\alpha = ker(X(T) - \alpha I) \subset C^n$$

$$G_\alpha = gen\ ker\ (X(T) - \alpha I) \subset C^n$$

where $gen\ ker\ B = ker\ B^m$ for large m is the generalized kernel of a matrix.

Given $\sigma > 0$ define

$$\mathcal{E}_\sigma = Re \bigoplus_{|\alpha|=\sigma} E_\alpha$$

$$\mathcal{G}_\sigma = Re \bigoplus_{|\alpha|=\sigma} G_\alpha$$

the real parts of the spans.

Theorem 2.2 (Mallet-Paret and Smith,[19]) *Let us consider (15) and assume first that $\Delta = -1$. If the dimension $n = 2b$ then the norms $\sigma = |\alpha|$ of the characteristic multipliers satisfy*

$$\sigma_1 \geq \sigma_2 > \sigma_3 \geq \sigma_4 > \ldots > \sigma_{n-1} \geq \sigma_n \tag{16}$$

and N takes values

$$N = 2h - 1 \ \text{on}\ \mathcal{G}_{\sigma_{2h-1}} + \mathcal{G}_{\sigma_{2h}} \qquad h = 1, 2, \ldots, b. \tag{17}$$

If $n = 2b - 1$ than one has

$$\sigma_1 \geq \sigma_2 > \sigma_3 \geq \sigma_4 > \ldots > \sigma_{n-2} \geq \sigma_{n-1} > \sigma_n$$

with $N = n$ on $\mathcal{G}_{\sigma_n}$ in addition to (17).

Now assume that $\Delta = 1$. *If* $n = 2b - 1$ *then*

$$\sigma_1 > \sigma_2 \geq \sigma_3 > \sigma_4 \geq \ldots > \sigma_{n-1} \geq \sigma_n$$

and

$$\begin{aligned} N &= 0 \text{ on } \mathcal{G}_{\sigma_1} \\ N &= 2h \text{ on } \mathcal{G}_{\sigma_{2h}} + \mathcal{G}_{\sigma_{2h+1}}, h = 1, \ldots, b-1. \end{aligned}$$

If $n = 2b$ *then*

$$\sigma_1 > \sigma_2 \geq \sigma_3 > \sigma_4 \geq \ldots \geq \sigma_{n-1} > \sigma_n$$

and $N = 2b$ *on* $\mathcal{G}_{\sigma_{2b}}$.

Let γ be a hyperbolic periodic orbit of $\mathcal{MCFS}$. Then the linear variational system is a monotone cyclic feedback system and has the form (15) and Theorem 2.2 applies.

Lemma 2.3 *Let* γ *be a hyperbolic periodic orbit of* $\mathcal{MCFS}$ *with a* k*-dimensional unstable manifold* W^u *and consider the linear variational system (15) along the solution* $\gamma(t)$. *We will use the notation from Theorem 2.2.*

If $\Delta = -1$ *and* k *is odd or* $\Delta = 1$ *and* k *is even then*

$$y \in TW^u_\Gamma \text{ implies } N(y) \leq k$$

$$y \in TW^s \text{ implies } N(y) \geq k + 2.$$

If $\Delta = -1$ *and* k *is even or* $\Delta = 1$ *and* k *is odd then*

$$y \in TW^u \text{ implies } N(y) \leq k - 1$$

$$y \in TW^s_\Gamma \text{ implies } N(y) \geq k + 1.$$

Remark 2.4 Here the notation $N(y) \geq l$ means that either $N(y)$ is defined and is equal to some integer which is greater or equal to l, or $N(y)$ is not defined, but y lies between the regions with $N = k_1$ and $N = k_2$ (i.e $N(y^+) = k_2$, $N(y^-) = k_1$) where $k_2 < k_1$ and $k_i \geq l$.

Proof. Note that since k is the dimension of an unstable manifold, the multiplier $\sigma_{k+1} = 1$. The different position of an index k in the inequalities in Theorem 2.2 under various combinations of $k \in \{\text{odd, even}\}$ and $\Delta = \pm 1$ is the reason, why the statements about the stable and unstable manifolds are different.

We prove the case $\Delta = -1$ and k odd, the other cases being analogous. Observe, that $x \in TW^s$ implies that the trajectory through the point x converge in the forward time to the orbit γ under the linear flow.

By (16) x approaches γ tangent to the eigenspace $\mathcal{G}_{\sigma_{k+2}} \oplus \mathcal{G}_{\sigma_{k+3}}$. Since the value of N is $k+2$ on $\mathcal{G}_{\sigma_{k+2}} \oplus \mathcal{G}_{\sigma_{k+3}}$ and the set on which $N(\cdot) = k+2$ is open, there exists a t_x such that for all $t \geq t_x$

$$N(x \cdot t) = k + 2.$$

Since the functional N is non-increasing along trajectories, this proves the second part of the Lemma for $y \in TW^s$.

Looking at convergence to γ as $t \to -\infty$ one shows, in a similar way, that for $y \in TW^u_\Gamma$ $N(y) \leq k$. □

Proof of Theorem 1.9. We consider the case $\Delta = -1$ and k is odd, the other cases being analogous.

Assume that $u \in (W^s \setminus G) \cap (W^u \setminus G)$ and let $u(t)$ be a solution with $u(0) = u$. Let $G(t)$ be a periodic solution defining the orbit G and let

$$z(t) = \frac{u(t) - G(t)}{|u(t) - G(t)|}.$$

By assumption $z(0) \neq 0$. Since G is hyperbolic, as $t \to -\infty$, $z(t) \to z_0$ and $z_0 \in TW^u_G$. Since N is finite and locally constant there is a $T < 0$ such that if $t < T$ then

$$z(t) \in \mathcal{N} \text{ and } N(z(t)) = N(z_0) \leq k$$

by Lemma 2.3.

Again by the hyperbolicity of G, as $t \to \infty$, $\bar{z}(t) \to \bar{z}_0$ and $\bar{z}_0 \in TW^s_G$. Lemma 2.3 implies that $N(z(t)) \geq k + 2$ for $t > \bar{T}$.

Thus $N(z(t)) \leq k$ for $t \to -\infty$ and $N(z(t)) \geq k + 2$ for $t \to \infty$ which contradicts Theorem 2.1. □

2.2 Periodic orbit γ.

In this subsection we further investigate properties of periodic orbits of $\mathcal{MCFS}$. We shall assume for this subsection that γ is a hyperbolic periodic orbit of $\mathcal{MCFS}$ with 1-dimensional unstable manifold. In the notation used in Lemma 2.3 this implies $\Delta = -1$ and $k = 1$. We also assume without loss of generality that $\delta_1 = -1$ and $\delta_i = 1$ for $i = 2, \ldots, n$. The following two lemmas, which are consequences of the existence of the discrete Ljapunov function, will be applied to a periodic orbit γ of the system (8), which is a $\mathcal{MCFS}$.

Let us reparameterize time so that $\gamma(0) = M$ (see Definition 1.10).

Lemma 2.5 $[\ddot{\gamma}(0)]_n < 0$.

Proof. Since $\gamma(t)$ is a solution of a $\mathcal{MCFS}$, the linear variational system along $\gamma(t)$, which is of the form (15), is a $\mathcal{MCFS}$. Observe that $\dot{\gamma}(t)$ a periodic solution of (15). Therefore $\dot{\gamma}(t) \in \mathcal{N}$ for all t and by Theorem 2.1.5 $([\dot{\gamma}(t)]_n, [\ddot{\gamma}(t)]_n) \neq (0, 0)$. Since $M = \gamma(0)$ is a maximum for the n-th coordinate along $\gamma(t)$ we have that $[\dot{\gamma}(0)]_n = 0$ and hence $[\ddot{\gamma}(0)]_n \neq 0$. Consequently,

$$[\ddot{\gamma}(0)]_n < 0.$$

□

Lemma 2.6 $[\dot{\gamma}(0)]_i < 0$ *for* $i \neq n$.

Proof. Let us consider again the linear variational system (15) around $\gamma(t)$. Since γ has a 1-dimensional unstable manifold, there is one characteristic multiplier σ_1 with $|\sigma_1| > 1$. Since $\Delta = -1$ and $k = 1$, by Theorem 2.2 we have $N(\dot\gamma(t)) = 1$. Observe that $[\dot\gamma(0)]_n = 0$ by the definition of the point M. Since $\dot\gamma(t)$ is a periodic solution of (**??**), it follows from Theorem 2.1.3, that $\dot\gamma(t) \in \mathcal{N}$ for all t. In particular, $\dot\gamma(0) \in \mathcal{N}$ and so by the definition of N

$$\delta_1\delta_n[\dot\gamma(0)]_{n-1}[\dot\gamma(0)]_1 < 0.$$

Since $\delta_1 = -1$ and $\delta_n = 1$ we get $[\dot\gamma(0)]_{n-1}[\dot\gamma(0)]_1 > 0$. By Lemma 2.5

$$0 > [\ddot\gamma(0)]_n = -a_n[\dot\gamma(0)]_n + b_n[\dot\gamma(0)]_{n-1}$$

where a_n and b_n are coefficients in (8). Since $b_n > 0$ and $[\dot\gamma(0)]_n = 0$ we get $[\dot\gamma(0)]_{n-1} < 0$ and therefore $[\dot\gamma(0)]_1 < 0$.

We want to prove that

$$[\dot\gamma(0)]_i < 0 \text{ for } i \neq n.$$

Assume to the contrary that there is an index j such that $[\dot\gamma(0)]_i < 0$ for $n-1 \geq i > j$ and either $[\dot\gamma(0)]_j = 0$ or $[\dot\gamma(0)]_j > 0$ (see Figure 4).

Let us discuss the later case. Since $[\ddot\gamma(0)]_n < 0$ we have $[\dot\gamma(t)]_n < 0$ for small positive t. But then for small positive t we have $N(\dot\gamma(t)) \geq 3$ because $\delta_1 = -1$ imply there will be one contribution to the number N between x_n and x_1 since $\delta_1[\dot\gamma(t)]_1[\dot\gamma(t)]_n < 0$; one contribution between indices j and $j+1$ ($\delta_{j+1}[\dot\gamma(t)]_{j+1}[\dot\gamma(t)]_j < 0$) and one contribution between indices j and 1 because $[\dot\gamma(t)]_j > 0$, $[\dot\gamma(t)]_1 < 0$ and all $\delta_i = 1, \ 1 < i \leq j$.

In the former case, since $\dot\gamma(t) \in \mathcal{N}$ for all t, it follows from equation (8) that $[\ddot\gamma(0)]_j = B_j[\dot\gamma(0)]_{j-1} < 0$ for some constant B_j. Since $j > 1$ we have that $B_j = b_j > 0$. Therefore $[\ddot\gamma(0)]_j < 0$ and so for $t > 0, \ \ t << 1$ we have $[\dot\gamma(t)]_j < 0$ and we are in the later case with $j = j+1$ and t small.

We established a contradiction, which proves the lemma. □

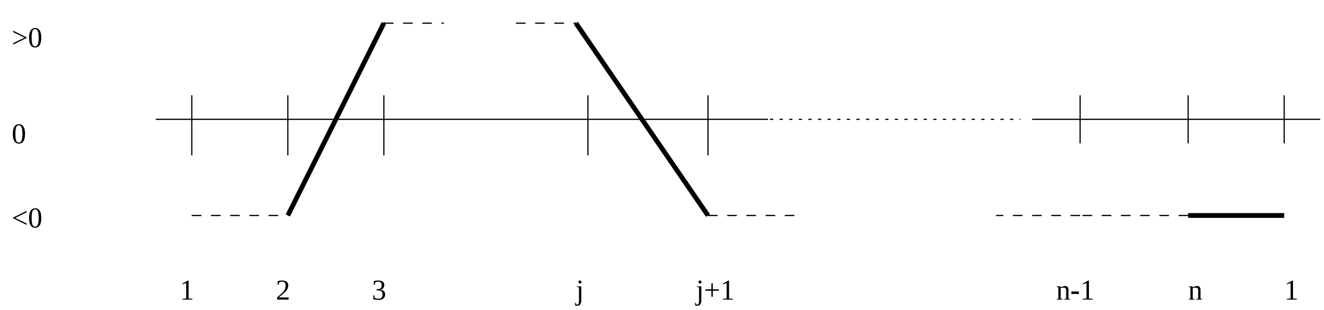


Figure 4: Full lines denote pairs which contribute to $N(\dot\gamma(t))$ and numbers indicate components of $\dot\gamma(t)$.

2.3 Topological crossing of manifolds.

We will follow the exposition of K. Burns and H. Weiss [2], but we will restrict our considerations to the case when the underlying manifold is orientable. For the general treatment the reader should consult the original paper.

Let Y and Z be compact oriented manifolds with boundary, smoothly embedded in an oriented manifold X. Suppose that

$$\partial Y \cap Z = \emptyset = \partial Z \cap Y \tag{18}$$

and $dim\, Y + dim\, Z = dim\, X$. We wish to define the (oriented) *intersection number of* Y *with* Z, which we denote by $\#(Y, Z)$.

Let us recall the definition in the case when Y and Z are without boundary ([10]). The orientations of X, Y and Z allow one to assign an intersection number of 1 or -1 to each transversal intersection of Y and Z. If Y and Z are transverse, then there is a finite number of intersection points and $\#(Y, Z)$ is the sum of the intersection numbers at all intersection points. It is easily shown that if Y' is homotopic to Y, Z' homotopic to Z, Y transversal to Z, Y' transversal to Z', then $\#(Y, Z) = \#(Y', Z')$.

One defines the intersection number in the general case by first performing a homotopy to make the submanifolds transversal, and then using the definition for the transversal case.

This procedure can be carried over to the case where the submanifolds have a nonempty boundary, with one caveat. The invariance of the intersection number breaks down, if the boundary of one of the submanifolds intersects the other manifold i.e. when (18) is violated during the homotopy. However, one can make submanifolds transversal with a homotopy, which moves the points less then any prescribed distance. Thus we add to the definition a condition, that the homotopy, used to make manifolds transversal, should move points by less then ϵ, where

$$0 < \epsilon < min(dist(\partial Y, Z), dist(\partial Z, Y)).$$

Definition 2.7 Let W and W' be smoothly immersed oriented submanifolds of X with complementary dimension. We say that W and W' *cross topologically* if there are compact embedded submanifolds with boundary, $V \subset W$ and $V' \subset W'$, such that

1. $dim\, V = dim\, W$ and $dim\, V' = dim\, W'$,
2. $\partial V \cap V' = \emptyset = \partial V' \cap V$
3. $\#(V, V') \neq 0$.

The importance of the topological crossing comes from the following Theorem.

Theorem 2.8 (Burns, Weiss [2]; Gedeon, Mischaikow, McCord [8])

Let M be a manifold and $f : M \to M$ be a diffeomorphism with a hyperbolic fixed point p. Assume that $W^s \setminus \{p\}$ and $W^u \setminus \{p\}$ have a topological crossing.

Then there is an invariant set S for f, surjective map ρ and an integer d such that the following diagram commutes

$$\begin{array}{ccc} S & \xrightarrow{f^d} & S \\ \rho \downarrow & & \downarrow \rho \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array}$$

i.e. f^d *on* S *is semi–conjugate to a shift* (Σ, σ).

The following Theorem shows a particular way, how to prove that two manifolds have a topological crossing and hence points out the applications of Theorem 2.8.

Theorem 2.9 *Let* $Q \subset \mathbf{R}^n$ *be a closed, connected set with nonempty interior and let* $V \subset Q$ *be a closed set such that*

$$Q \setminus V = Q^+ \cup Q^-,$$
$$Q^+ \cap Q^- = \emptyset.$$

Assume that there is an n - 1 *dimensional oriented manifold* B *with the boundary,* $B \subset V$, *which divides* Q *into two parts. Furthermore, let* $A \subset \text{int}Q$ *be a one-dimensional manifold with the boundary* $\partial A = \{A_1, A_2\}$, *such that* $A_1 \in Q^+$ *and* $A_2 \in Q^-$ *(see Figure 5).*

Then B *and* A *cross topologically.*

Proof. Note that condition 1 from the definition above is satisfied for B and A.

Since B divides Q into two parts, $\partial B \subset \partial Q$ and since $A \subset \text{int}\, Q$ we have $\partial B \cap A = \emptyset$. Observe further, that $\partial A \cap V = \emptyset$ and since $B \subset V$ we have $\partial A \cap B = \emptyset$. Thus the condition 2 in the definition of the topological crossing is satisfied for A and B.

Since $B \subset V$ and A is compact, we can construct a homotopy, which makes A and B transversal, in such a way that it will move only points in V which are a positive distance from ∂V. Since $\partial B \subset \partial Q$ and $B \subset V$, we have $\partial B \subset \partial V$. Now $\partial A \notin V$ by assumption, and so the homotopy will not move ∂B and ∂A. Thus condition 2 is satisfied throughout the homotopy.

Since A is one-dimensional, and the endpoints are in Q^+ and Q^-, a simple count shows that

$$\#(A, B) = \pm 1 \neq 0.$$

□

We will also need the following trivial observation.

Lemma 2.10 *If* $A \cap B = \emptyset$ *then* $\#(A, B) = 0$.

3 Main results.

We start the section with the following Theorem.

Theorem 3.1 *Assume that the system (8) in* $\mathbf{R}^n$ *admits a hyperbolic periodic orbit* γ *with one-dimensional unstable manifold. Fix* L *such that* $0 < L < f(M_n)$ *and choose a two dimensional family* $\mathcal{G} : (\delta, \eta) \to \hat{g}_{\delta,\eta}$ *of functions of the form (10).*

Consider any one dimensional family $\mathcal{F} \subset \mathcal{G}$ *parameterized by a continuous curve of the form* $(\delta, \eta(\delta))$ *in the neighborhood of* $(0, 0)$ *and*

$$\eta(\delta) = O(\delta^q), \qquad q > 2 \qquad \text{as } \delta \to 0. \tag{19}$$

If either

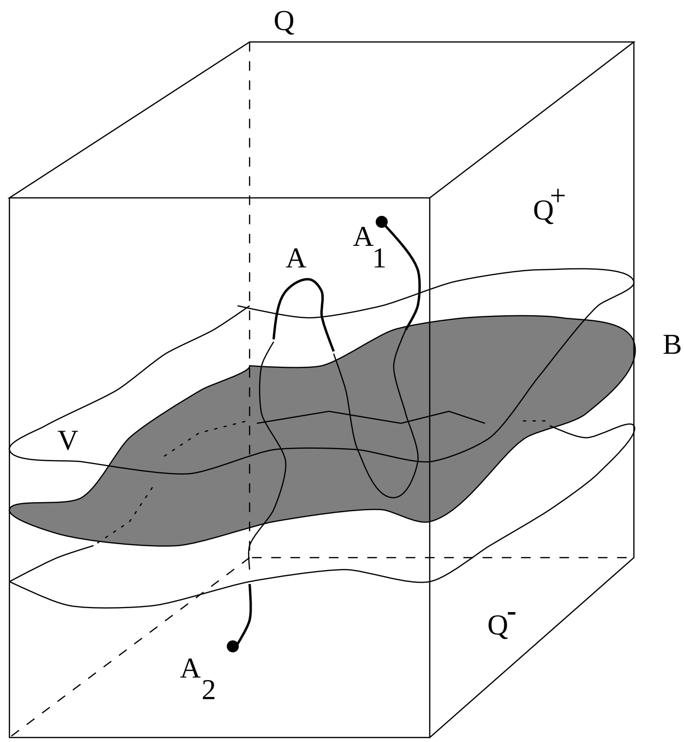


Figure 5: A topological crossing of A and B .

- $n = 3$ *or*
- $(C+2)\rho^u < \rho^s$ *with* $C = \sqrt{(1 - \frac{\alpha \cdot m}{[\alpha]_1})^2 + \sum_{i=2}^n m_i^2}$

then there exist $\bar{\delta} = \bar{\delta}(\mathcal{F}, \mathcal{G}) > 0$ *and a set* $N \subset H$ *with the following properties.*

1. *For any* $\delta \leq \bar{\delta}$ *the system (9) with* $\hat{g}_{\delta,\eta(\delta)} \in \mathcal{F}$ *admits* N *as a Poincarè section with a Poincarè map* $\Pi_{\hat{g}_{\delta,\eta(\delta)}}$.
2. *Let* p *be a fixed point of* $\Pi_{\hat{g}_{\delta,\eta(\delta)}}$ *corresponding to the periodic orbit* γ*. Then for all* $\delta \leq \bar{\delta}$ *the stable and unstable manifolds of* p *under* $\Pi_{\hat{g}_{\delta,\eta(\delta)}}$ *have a topological crossing for* $\hat{g}_{\delta,\eta(\delta)} \in \mathcal{F}$.

Note, that the Theorem 1.11 is a corollary of the Theorem 3.1 and the Theorem 2.8. Thus in order to prove Theorem 1.11 it suffices to prove the Theorem 3.1.

3.1 Outline of the proof of Theorem 3.1

In this section we outline the proof of Theorem 3.1. We fix L such that $0 < L < f(M_n)$ and consider a family $\mathcal{F} \subset \mathcal{G}$ as in the Theorem 3.1, parameterized by the curve $(\delta, \eta(\delta))$.

To indicate the fact that we will take $\hat{g}_{\delta,\eta}$ from a one dimensional family $\mathcal{F}$ we will use the notation $\hat{g}_\delta \in \mathcal{F}$, since by assumption the family $\mathcal{F}$ can be parameterized by δ.

We let $\delta \to 0$ and our goal is to construct neighborhoods Q_δ and V_δ such that the assumptions of Theorem 2.9 will be satisfied for sufficiently small δ with B_δ some subset of the stable manifold and A_δ some subset of the unstable manifold of the periodic orbit γ in the system (9) with $\hat{g}_\delta$. In other words, we will change the nonlinearly $\hat{g}_\delta$ along the family $\mathcal{F}$ as $\delta \to 0$ and for sufficiently small $\bar{\delta}$ the Theorem 3.1 will hold.

We will first introduce some notation.

We denote by $F(x)$ the vector field given by (8) and by $\hat{G}_\delta(x)$ a vector field given by (9) with $\hat{g}_\delta$. Further, let $\Phi(x,t)$ be the flow generated by the vector field $F(x)$ and let $\hat{\Phi}_\delta(x,t)$ be the flow generated by $\hat{G}_\delta(x)$.

Let W^u and W^s be the unstable and the stable manifolds of γ under Φ and let $\hat{W}^s_\delta, \hat{W}^u_\delta$ be the stable and unstable manifolds of γ under $\hat{\Phi}_\delta(x,t)$. Also recall, that in the Introduction we have reparameterized the time along γ so that $\gamma(0) = M$ where $[M]_n$ is the maximum in the n-th coordinate along γ.

Recall, that $H := \{x \in \mathbf{R}^n \mid a_n x_n = b_n x_{n-1}\}$ is the hyperplane where $[\dot{x}]_n = 0$. We will show in Lemma 3.3, that there is a subset $N \subset H$ which serves as a Poincaré section for Φ and $\hat{\Phi}_\delta$.

Observe that for $x \in (-\infty, M_n + \delta]$ $f(x) = \hat{g}_\delta(x)$ for all δ. This observation leads to the following definition.

Definition 3.2 *Let*

$$Z_\delta = \{x \in R^n \mid [x]_n \leq M_n + \delta\}$$

be the set where $F(x) \equiv \hat{G}_\delta(x)$ *and let*

$$J_\delta = \{x \in R^n \mid [x]_n = M_n + \delta\}$$

be its boundary. The set $J_\delta = J_\delta^- \cup J_\delta^+$ *decomposes into two sets such that if* $x \in J_\delta^-$ *then* $[\dot{x}]_n \leq 0$ *and if* $x \in J_\delta^+$ *then* $[\dot{x}]_n \geq 0$. *Observe, that the boundary between* J_δ^- *and* J_δ^+ *is* $H \cap J_\delta$.

Note that since $\gamma(t) \subset Z_\delta$, $\gamma(t)$ is a periodic solution of the flow $\hat{\Phi}_\delta$ for any δ.

We will outline the main steps in the proof of Theorem 3.1 and illustrate them by pictures for n=3. The proof of second part of the Theorem 3.1 is more involved and so the main effort will be directed in that direction.

1. As a first step we prove Theorem 3.1.1 and then establish, that for sufficiently small δ

$$W^u \cap (\mathbf{R}^n \setminus Z_\delta) \neq \emptyset, \qquad W^s \cap (\mathbf{R}^n \setminus Z_\delta) \neq \emptyset. \tag{20}$$

   Observe that since f is monotone it follows from the Theorem 1.9, that

$$(W^s \setminus \gamma) \cap (W^u \setminus \gamma) = \emptyset.$$

   However, since we want to prove that

$$(\hat{W}^s_\delta \setminus \gamma) \cap (\hat{W}^u_\delta \setminus \gamma) \neq \emptyset$$

   (20) is a necessary prerequisite, since the flows Φ and Φ_δ differ only in $\mathbf{R}^n \setminus Z_\delta$.

   In the proof we use in an essential way the results from chapter 2.1 about the linear variational equation along $\gamma(t)$.

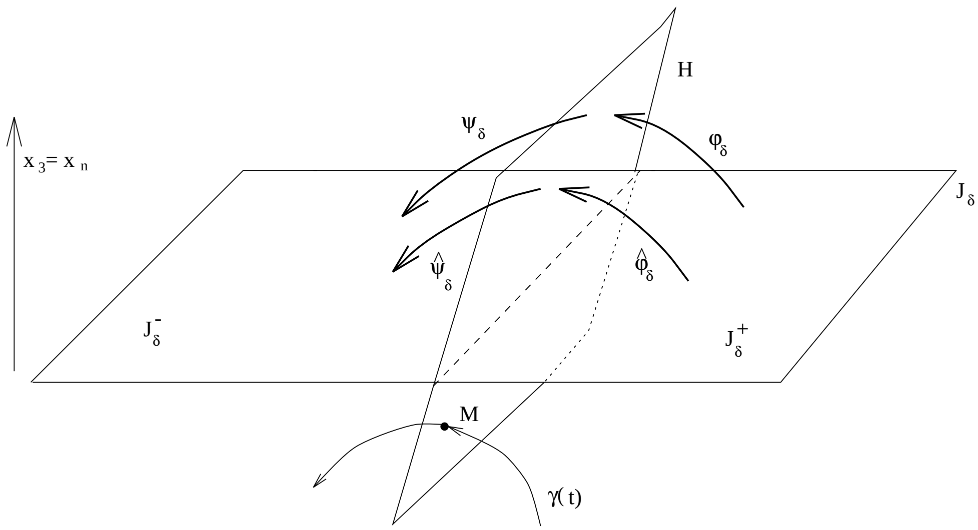


Figure 6: Flow defined map $\varphi_\delta, \hat\varphi_\delta, \psi_\delta$ and $\hat\psi_\delta$.

2. The proof of Theorem 3.1 will consist of two major steps. The first involves the estimate on how much the trajectories under two flows Φ and $\hat\Phi_\delta$, starting from the same initial data on J^+_δ, separate before reaching the hyperplane H. The second step involves the estimate on how much the trajectories separate, if they start from the point on H, before hitting J^-_δ.

   In order to capture the separation we shall consider flow defined maps $\varphi_\delta, \hat\varphi_\delta : J^+_\delta \to H$ defined by the flows Φ and $\hat\Phi_\delta$, respectively, and, for the second step, $\psi_\delta, \hat\psi_\delta : H \to J^-_\delta$ defined again by the flows Φ and $\hat\Phi_\delta$ respectively (Figure 6). The subscript indicates the fact, that since the hyperplane J_δ and the flows $\hat\Phi_\delta$ change with δ, we need to consider the families of maps $\varphi_\delta, \hat\varphi_\delta, \psi_\delta$ and $\hat\psi_\delta$, parameterized by δ.

3. In order to show that the stable and unstable manifolds under the flow $\hat\Phi_\delta$ intersect, we need to understand how the quantities

$$\varphi_\delta(x_\delta) - \hat\varphi_\delta(x_\delta)$$

   for some $x_\delta \in J^+_\delta$ and

$$\psi_\delta(x_\delta) - \hat\psi_\delta(x_\delta)$$

   for $y_\delta \in H$ approach 0 as $\delta \to 0$. This question makes sense only if we specify the choice of x_δ and y_δ since for different δ these are different points.

   The set $\{x_\delta\}$ will be called a *family* if there is a $\delta' > 0$ such that for each $\delta \in (0, \delta']$ there is unique $x_\delta \in \{x_\delta\}_{(0,\delta']}$. In other words a family is not necessarily a continuous function $\delta \to x_\delta$ with the domain $(0, \delta]$. To simplify the notation we shall drop the brackets and use notation x_δ for the family $\{x_\delta\}$. We shall consider both continuous families $x_\delta \in J^+_\delta$ and families $y_\delta \in H$. It is intuitively clear, that the longer time the trajectory with initial value $x_\delta \in J^+_\delta$ stays out of Z_δ, the more the trajectories under Φ and $\hat\Phi_\delta$ differ. This time can be characterized by the maximal height above the J_δ, which the trajectory starting from a point achieves under Φ; the higher it gets the longer it stays out of Z_δ. Given a family $x_\delta \in J_\delta^+$ for $\delta \in (0, \delta']$ we will be interested in how quickly the maximal achieved height above J_δ goes to 0 as $\delta \to 0$.

This brings us to the concept of a p- and $(p,+)$-family. A p-family is a continuous function $y : [0, \delta'] \to H$, $\delta \mapsto y_\delta$ for some δ' with the property that the height above J_δ goes to zero as δ^p, i.e., $[y_\delta]_n - M_n - \delta = O(\delta^p)$ as $\delta \to 0$. The $(p,+)$-family is a continuous function $x : [0, \delta''] \to J_\delta^+$, $\delta \mapsto x_\delta$ for some δ'' such that the intersection of the forward trajectory with H is a p-family. In other words, x_δ is a $(p,+)$-family if $\varphi_\delta(x_\delta)$ is a p-family. The $+$ indicate the fact that the family is a subset of J_δ^+.

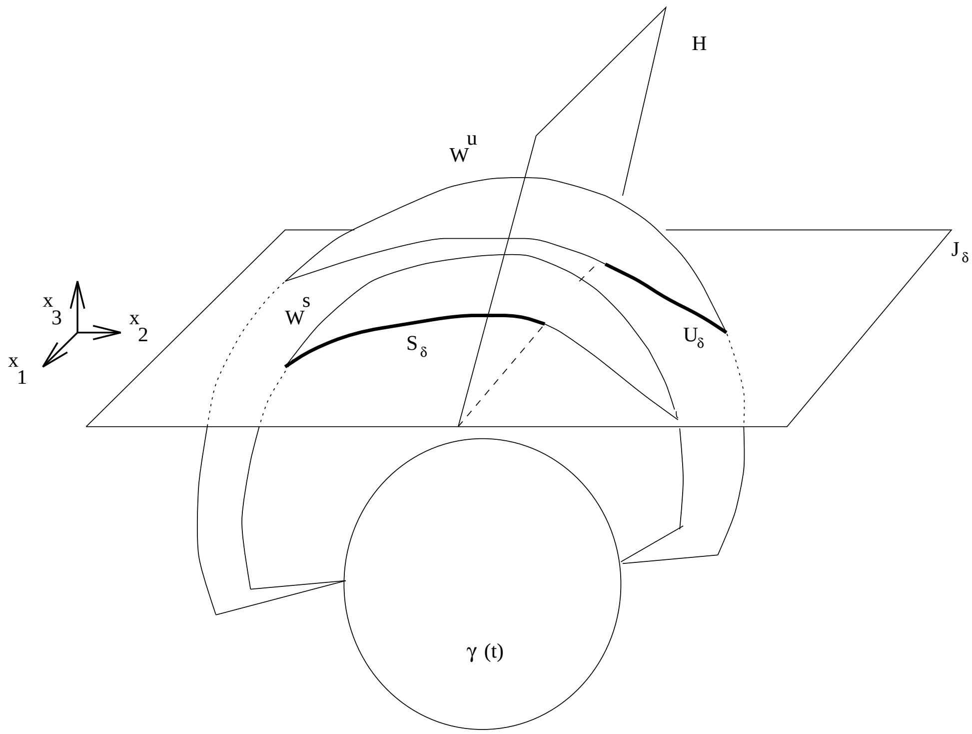


Figure 7: Sets $S_\delta, U_\delta \subset J_\delta$.

4. The results, obtained in step 1, give us some information about the location of the manifolds W^s and W^u. To gain some information about the manifolds $\hat{W}_\delta^s$ and $\hat{W}_\delta^u$ we identify subsets of W^s and W^u which are also subsets of $\hat{W}_\delta^s$ and $\hat{W}_\delta^u$ respectively.

   Let

$$S_\delta := \{x \in J_\delta^- \cap W^s \mid \Phi(x,t) \in Z_\delta \text{ for } t \geq 0\}$$
$$U_\delta := \{x \in J_\delta^+ \cap W^u \mid \Phi(x,t) \in Z_\delta \text{ for } t \leq 0\}.$$

   Observe that S_δ and U_δ have the desired property

$$S_\delta \subset J_\delta^- \cap \hat{W}_\delta^s$$
$$U_\delta \subset J_\delta^+ \cap \hat{W}_\delta^u$$

   since $F(x) = \hat{G}_\delta(x)$ in Z_δ. Using the hyperbolicity of γ we show that $S_\delta \neq \emptyset$ and $U_\delta \neq \emptyset$ for δ sufficiently small (Figure 7).

The final goal of the proof of Theorem 3.1 is to show that for sufficiently small δ the set $\hat{\psi}_\delta \circ \hat{\varphi}_\delta(U_\delta)$ will intersect S_δ in a significant way, i.e. these two sets will have a topological intersection. Since $S_\delta \subset \hat{W}^s$ and both $\hat{\psi}_\delta$ and $\hat{\varphi}_\delta$ are $\hat{\Phi}_\delta$ defined maps, $\hat{\psi}_\delta \circ \hat{\varphi}_\delta(U_\delta) \subset \hat{W}^u$ and so this intersection will be an intersection of $\hat{W}^u$ and $\hat{W}^s$.

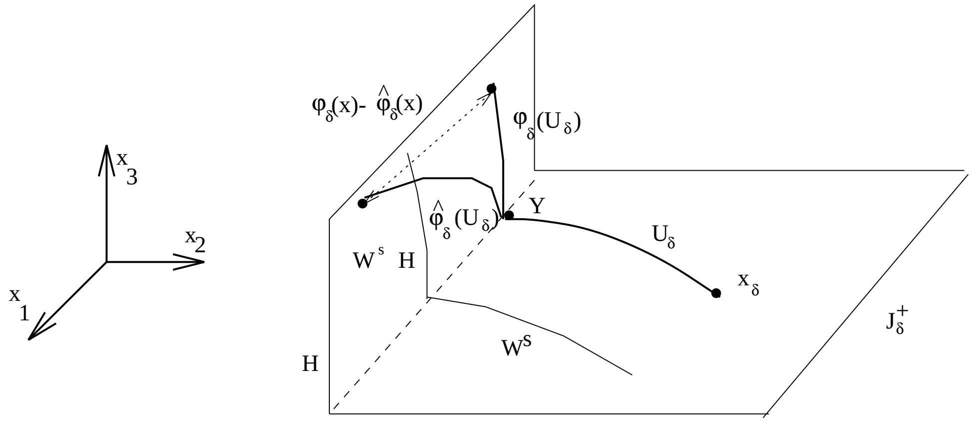


Figure 8: The difference $\varphi_\delta(x_\delta) - \hat{\varphi}_\delta(x_\delta)$.

5. The proof that S_δ and $\hat{\psi}_\delta(\hat{\varphi}_\delta(U_\delta))$ intersect involve two steps as was mentioned in the step 2. In the first step we show that $\hat{\varphi}_\delta(U_\delta) \cap W^s \cap H \neq \emptyset$ for sufficiently small δ. Then we take a certain subset $\mathcal{V}$ of $\hat{\varphi}_\delta(U_\delta) \subset H$ and show that for $\hat{\psi}_\delta(\mathcal{V}) \subset \hat{\psi}_\delta(\hat{\varphi}_\delta(U_\delta))$ we have
$$\hat{\psi}_\delta(\mathcal{V}) \cap S_\delta \neq \emptyset.$$
The reason why we need to apply this two step approach lies in the way S_δ is defined. We shall explain this point in the step 6 of this outline.

Here we want to outline the argument in the proof of the fact that
$$\hat{\varphi}_\delta(U_\delta) \cap W^s \cap H \neq \emptyset$$
for sufficiently small δ. The proof of the second step, which involves the set $\mathcal{V}$ and the the maps ψ_δ and $\hat{\psi}_\delta$, uses precisely the same idea.

We take a certain continuous family $x_\delta \in U_\delta$ and investigate how the quantity $\varphi_\delta(x_\delta) - \hat{\varphi}_\delta(x_\delta)$ approaches 0 as $\delta \to 0$.

If we compare solutions with the same initial data $x_\delta \in U_\delta$ under two flows Φ and $\hat{\Phi}_\delta$, it follows from the structure of the cyclic feedback system that the trajectories differ the most in the first coordinate. This is indicated in Figure 8. We will show that for a $(p,+)$-family $x_\delta \in U_\delta$
$$\varphi_\delta(x_\delta) - \hat{\varphi}_\delta(x_\delta) = \hat{c}(x_\delta)e_1 + \text{higher order terms} \tag{21}$$
where e_1 is a first standard unit vector and $\hat{c}(x_\delta)$ and is a constant for given $(p,+)$-family x_δ which, however, depends on p.

We introduce a vector α perpendicular to the hyperplane $TW^s_\Gamma(0)$ and try to show, that for some x_δ the projection of $\hat{\varphi}_\delta(x_\delta)$ onto α has the opposite sign as the projection

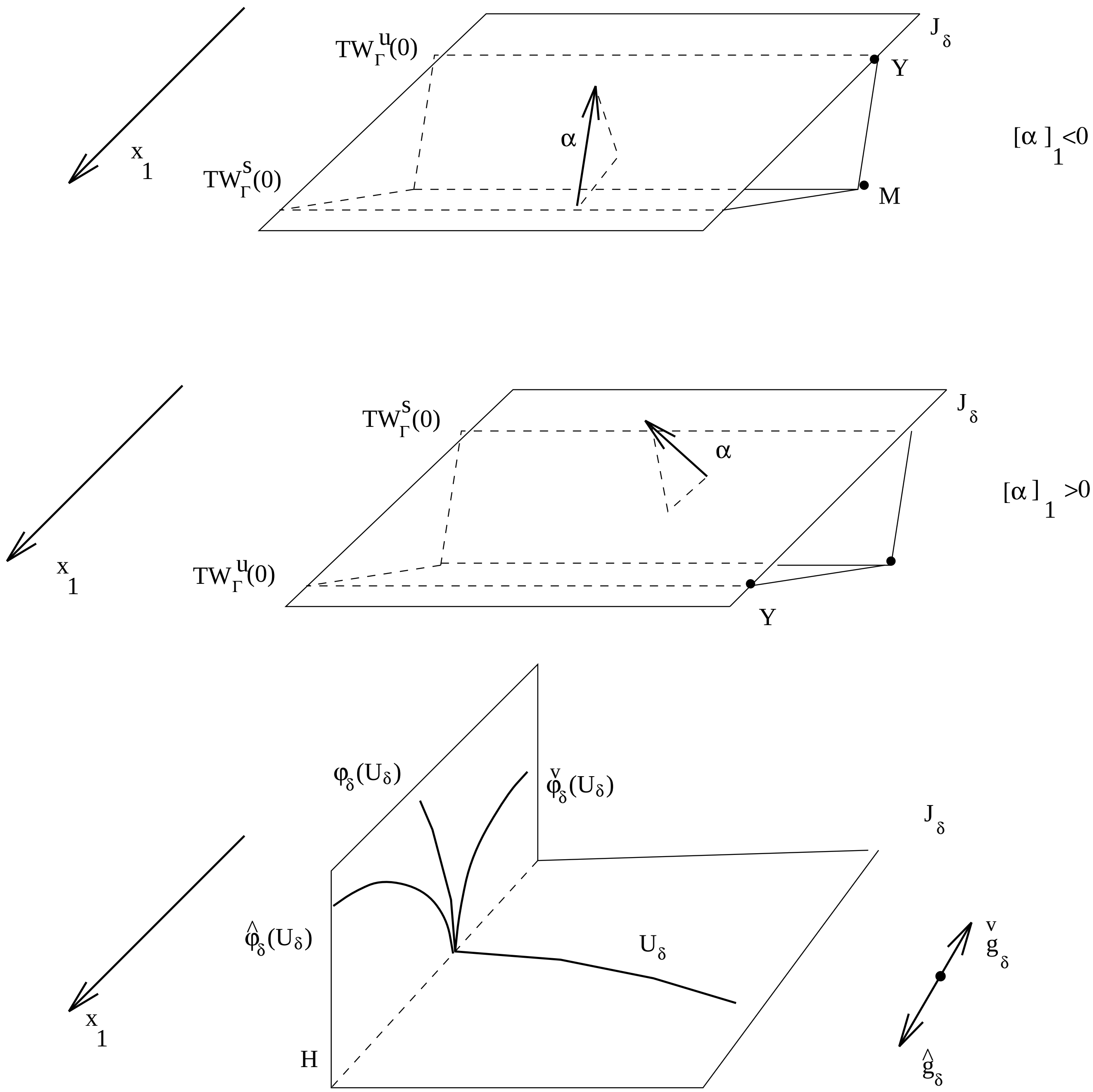


Figure 9: Two possibilities with $[\alpha]_1 > 0$ and $[\alpha]_1 < 0$ are depicted. The arrows on the bottom picture indicate the direction in which $\hat{\varphi}_\delta(U_\delta)$ vs. $\check{\varphi}_\delta(U_\delta)$ deviate from $\varphi_\delta(U_\delta)$.

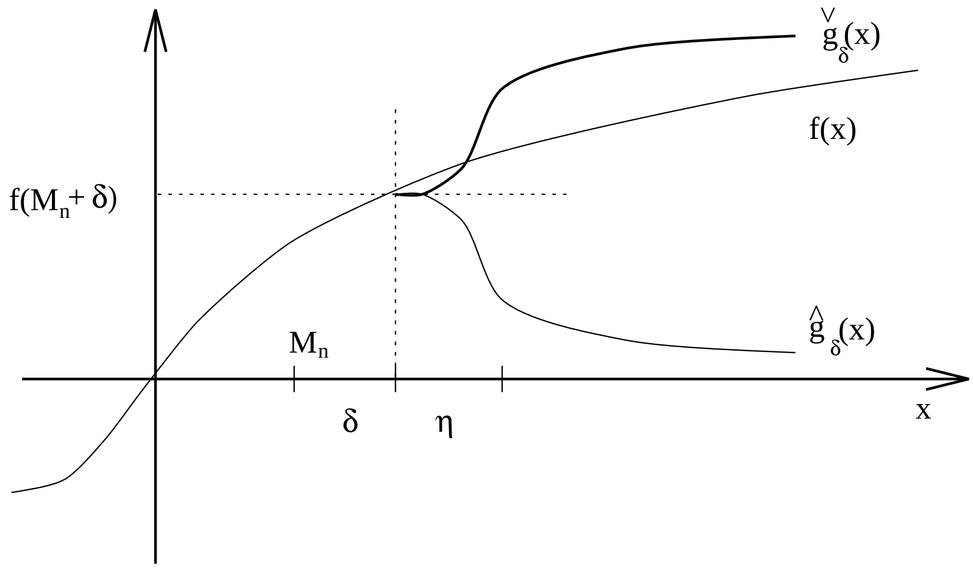


Figure 10: Function $\check{g}_\delta(x)$.

of the point $Y \subset H \cap U_\delta$. This will indicate that $\hat{\varphi}_\delta(U_\delta)$ intersect $TW^s_\Gamma(0)$ and hence suggest that $\hat{\varphi}_\delta(U_\delta) \cap H \cap W^s \neq \emptyset$.

First we fix the direction of α by requiring

$$\alpha \cdot (Y - M) > 0$$

for $Y \in H \cap U_\delta$. But since the change of the flow is mainly along the first coordinate we will need to know the sign of $[\alpha]_1$. We do not know how to determine this sign directly (Figure 9), since it depends on a mutual position of $TW^s_\Gamma(0)$ and $TW^u_\Gamma(0)$.

Instead we choose the following approach. To every function $\hat{g}_\delta \in \mathcal{F}$ we construct an auxiliary function $\check{g}_\delta$ in the following way

$$\begin{aligned} \check{g}_\delta(x) &= f(x) \text{ if } x \in (-\infty, M_n + \delta] \\ \check{g}_\delta(x) &= 2f(x) - \hat{g}_\delta(x) \text{ for } x \in (M_n + \delta, \infty) \end{aligned}$$

Observe, that $\check{g}_\delta$ is obtained from $\hat{g}_\delta$ by flipping the graph of $\hat{g}_\delta$ on $[M_n+\delta, \infty)$ about $y = f(x)$ and letting $\check{g}_\delta = f$ elsewhere. It is important to note that from this construction and the definition of the functions $\hat{g}_\delta$ it follows that $\check{g}_\delta(x)$ is monotonically increasing and so system (8) with $\check{g}_\delta$ instead of f is a $\mathcal{MCFS}$. We want to remark that our choice of the notation was motivated by the form of the graph of the functions $\hat{g}_\delta$ and $\check{g}_\delta$; since $\hat{g}_\delta$ has a hump we use a "hat" as a superscript and since $\check{g}_\delta$ is a flip of $\hat{g}_\delta$ we flipped the superscript and use a "check" in this case.

Let $\check{\Phi}_\delta$ be a flow generated by (8) with f replaced by $\check{g}_\delta$. We define the maps $\check{\psi}_\delta$ and $\check{\varphi}_\delta$ in an analogous way as the maps $\hat{\psi}_\delta$ and $\hat{\varphi}_\delta$ respectively, using the flow $\check{\Phi}_\delta$. We will show that

$$\varphi_\delta(x_\delta) - \check{\varphi}_\delta(x_\delta) = \check{c}(x_\delta)e_1 + \text{higher order terms} \tag{22}$$

for $x_\delta \in U_\delta$ in analogy with (21).

Observe that from the construction of $\check{g}_\delta(x)$ it follows that $f(x) - \check{g}_\delta(x) = -(f(x) - \hat{g}_\delta(x))$. This will imply that $|\check{c}(x_\delta)| = |\hat{c}(x_\delta)|$ but

$$\hat{c}(x_\delta) < 0 \qquad \text{and} \qquad \check{c}(x_\delta) > 0.$$

This together with (21) and (22) indicate the direction in which $\check{\varphi}_\delta(U_\delta)$ resp. $\hat{\varphi}_\delta(U_\delta)$ deviate from $\varphi_\delta(U_\delta)$ (Figure 9). There are two possibilities with $[\alpha]_1 < 0$ and $[\alpha]_1 > 0$ depicted on Figure 9. We will show that for a fixed family x_δ and sufficiently small δ the constants $|\check{c}(x_\delta)| = |\hat{c}(x_\delta)|$ are sufficiently large for the intersection to occur; however, as indicated in the figure 9, if $[\alpha]_1 > 0$ then the sets $\check{\varphi}_\delta(U_\delta)$ and $W^s \cap H$ will intersect and if $[\alpha]_1 > 0$ then the sets $\hat{\varphi}_\delta(U_\delta)$ and $W^s \cap H$ will intersect.

The punch line is: since $\check{g}_\delta$ is monotone, Theorem 1.9 rules out the second possibility.

To be able to fulfill this plan and state and discuss results which hold for both flows $\hat{\Phi}_\delta$ and $\check{\Phi}_\delta$, we introduce a notation Φ^*_δ. One should think of Φ^*_δ as denoting exactly one of the flows $\hat{\Phi}_\delta$, $\check{\Phi}_\delta$ but we do not know at the moment which one. This will become clear only in the last step of the proof. At that moment we will be able to go back and replace all $*$ signs in all statements by the $\hat{}$ sign and all the statements will be true. All statements, unless otherwise noted, involving Φ^* are true for both flows $\hat{\Phi}$ and $\check{\Phi}$. Hence we shall go through the argument using notation Φ^*_δ. In order to keep the notation digestible, we will consistently use signs "hat" $\hat{}$, "check" $\check{}$ and "star" $*$ to denote objects connected to flows $\hat{\Phi}_\delta$, $\check{\Phi}_\delta$ and Φ^*_δ respectively. More precisely, we denote by W^{s*}_δ and W^{u*}_δ the stable and unstable manifolds of γ under Φ^*_δ and we denote by $G^*_\delta(x)$ a vector field corresponding to Φ^*_δ. We shall use ψ^*_δ instead of both $\hat{\psi}_\delta$ and $\check{\psi}_\delta$ and φ^*_δ instead of both $\hat{\varphi}_\delta$ and $\check{\varphi}_\delta$. Also we denote by $c^*(x)$ both $\hat{c}(x)$ and $\check{c}(x)$.

6. After all this preparation we begin proving the Theorem 3.1.2 (Figure 11).

   In the first step we construct neighborhoods $Q_\delta \subset H$ and $V_\delta \subset Q_\delta$ for sufficiently small δ such that

   (a) $A_\delta \subset \varphi^*_\delta(U_\delta)$
   (b) $B_\delta \subset W^s \cap Q_\delta \subset V_\delta$

   and we show that all the assumptions of Theorem 2.9 are verified with $Q = Q_\delta$, $V = V_\delta$, $A = A_\delta$ and $B = B_\delta$. It follows, that $\varphi^*_\delta(U_\delta)$ and $W^s \cap Q_\delta$ have a topological crossing.

   This intersection is not, however, the intersection of W^{s*} and W^{u*}. The only information we have about W^{s*} is that the set $S_\delta \subset J^-_\delta$ is a part of W^{s*}. Thus we need to exhibit the intersection of $\psi^*_\delta(\varphi^*_\delta(U_\delta))$ and S_δ. The characterization of S_δ is given through the multiplier of a Poincaré map in the following way:

   $$\text{if } y \in H \cap W^s \text{ and } ||y - M|| \le \rho^s \delta$$

   then $\psi_\delta(y) \in S_\delta$. This is the reason for the two step approach: we want to exhibit the intersection between $\psi^*_\delta(\varphi^*_\delta(U_\delta))$ and S_δ in the hyperplane J^-_δ but the set S_δ is characterized through its preimage under the flow Φ in H.

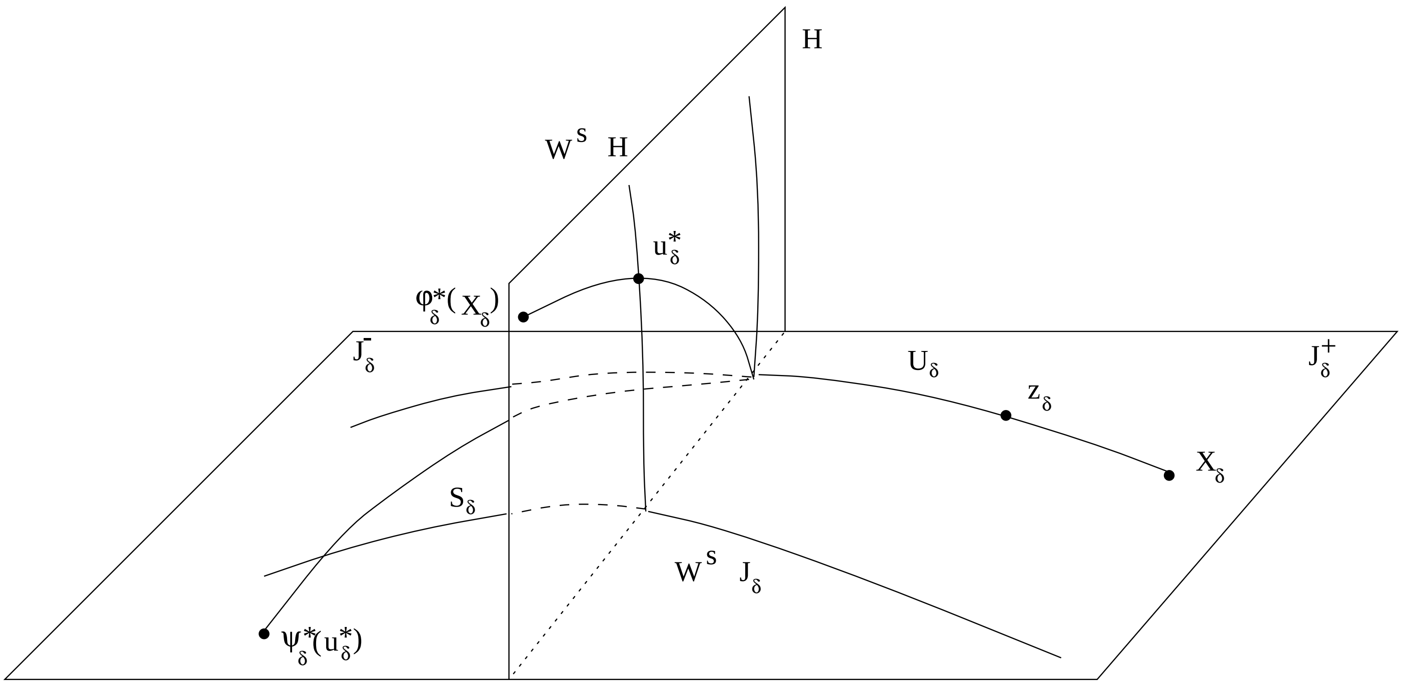


Figure 11: Two step approach in the proof of the intersection of W^{s*} and W^{u*}. For the sake of clarity the picture does not include neighborhoods Q_δ and V_δ.

After having established the intersection of $\varphi_\delta^*(U_\delta)$ and $W^s \cap H$, the second step will be to take a subset $\bar{U}_\delta$ of U_δ with one end point in $H \cap U_\delta$ and the other end point z_δ, with the property that $\varphi_\delta^*(z_\delta) = u_\delta^* \in W^s \cap \varphi_\delta^*(U_\delta) \cap H$. Then we look at the image of $\bar{U}_\delta$ under the composition $\psi_\delta^* \circ \varphi_\delta^*$. In this part we use the fact that

$$\psi_\delta(u_\delta^*) - \psi_\delta^*(u_\delta^*) = C^*(u_\delta^*)e_1 + \text{ higher order terms}$$

where $|C^*(u_\delta^*)|$ depends on the family u_δ^*. This is analogous to (21) and (22) for the difference $\varphi_\delta(x_\delta) - \varphi_\delta^*(x_\delta)$.

We build the sets $Q'_\delta \subset J_\delta^-$, $V'_\delta \subset Q'_\delta$ in a very similar way as in the first part and show that the image, $\mathcal{U}_\delta := \psi_\delta^* \circ \varphi_\delta^*(\bar{U}_\delta)$, has topological intersection with $W^s \cap J_\delta^-$.

However, this is not the end of the story, since we need to show that the sets $\mathcal{U}_\delta$ and $S_\delta \subset W^s \cap J_\delta^-$ have a topological intersection. Here we use the characterization of S_δ through its preimage in H.

We show that $||u_\delta^* - M|| \leq \rho^u\delta +$ higher order terms, and show that for all x_δ in $\varphi_\delta^*(\bar{U}_\delta)$ we have an estimate

$$||\varphi_\delta^*(x_\delta) - M|| \leq (C+2)\rho^u\delta + \text{ higher order terms }. \tag{23}$$

By the construction any point $z_\delta \in \mathcal{U}_\delta \cap W^s \cap J_\delta^-$ can be written as $\psi_\delta^*(\varphi_\delta^*(q_\delta))$ for some q_δ and $\varphi_\delta^*(q_\delta)$ satisfies (23).

Now we shall use the condition (14). If it is satisfied i.e. if $(C+2)\rho^u < \rho^s$ then for sufficiently small δ

$$||\varphi_\delta^*(q_\delta) - M|| \leq \rho^s\delta$$

and this will imply that $z_\delta = \psi^*_\delta(\varphi^*_\delta(x_\delta)) \in S_\delta$. Since this holds for arbitrary $z_\delta \in \mathcal{U}_\delta \cap W^s \cap J^-_\delta$ it shows that $\mathcal{U}_\delta \cap W^s \cap J^-_\delta \subset S_\delta$ and hence $\mathcal{U}_\delta \subset W^{*u}$ and $S_\delta \subset W^{*s}$ have topological intersection.

For $n = 3$ we show directly that $||\varphi^*_\delta(q_\delta) - M|| \leq \rho^s\delta$ hold and then we use the same argument to show that $\mathcal{U}_\delta$ and S_δ have topological intersection.

In this way we show that under condition (14) the sets W^{u*} and W^{s*} have topological intersection.

The conclusion of Theorem 3.1 follows after we pull back the intersection to the Poincarè section $N \subset H$ and apply the argument from step 5, concerning the direction of the change under $\hat{g}_\delta$ vs. $\check{g}_\delta$.

3.2 Step 1.

We begin by showing that there is a $N \subset H$, which is a Poincarè section for flows Φ^*_δ and Φ for all δ, where Φ^*_δ is generated by (9) with $\hat{g}_{\delta,\eta(\delta)} \in \mathcal{F}$. This will prove Theorem 3.1.1. Recall, that $M = \gamma(0) \in H$.

Lemma 3.3 *1. There is a neighborhood $N \subset H$ of $\gamma(0)$ such that the flow Φ and all the flows Φ^*_δ are transversal to H in N.*

2. $TW^s_\Gamma(0) \pitchfork H$, i.e. $TW^s_\Gamma(0)$ is transversal to H.

3. $TW^u_\Gamma(0) \pitchfork H$.

Proof. By Lemma 2.6 $[\dot{\gamma}(0)]_{n-1} \neq 0$. Since by definition $[\dot{\gamma}(0)]_n = 0$ and the hyperplane H is given by

$$H = \{x \in \mathbf{R}^n \mid x_n = \frac{b_n}{a_n} x_{n-1}\}$$

we see that $\dot{\gamma}(0) \notin H$. Since H is codimension 1, by continuity there is a neighborhood $N \subset H$ of $\gamma(0)$ such that Φ is transversal to H in N.

This means that if $y \in N$, then $F(y) \notin H$ and so

$$[F(y)]_n \neq \frac{b_n}{a_n}[F(y)]_{n-1}.$$

It follows from (8) that

$$[G^*_\delta(y)]_i = [F(y)]_i \qquad \text{for } i = 2, \ldots, n$$

and so we have

$$[G^*_\delta(y)]_n \neq \frac{b_n}{a_n}[G^*_\delta(y)]_{n-1}$$

which implies $G^*_\delta \notin H$. Hence Φ^*_δ is transversal to H in N. This finishes the proof of part 1.

Part 2 and 3 follow from the fact that $\dot{\gamma}(0) \in TW^s_\Gamma(0) \cap TW^u_\Gamma(0)$. □

The following lemma shows that $W^u \cap (\mathbf{R}^n \setminus Z_\delta) \neq \emptyset$ and, consequently, $W^u \neq W^{u*}_\delta$ in $(\mathbf{R}^n \setminus Z_\delta)$.

Lemma 3.4 *The set* $TW^u_\Gamma(0) \cap H = \{x \in \mathbf{R}^n \mid x = M + sm, s \in [-\infty, \infty)\}$ *is a line where the unit vector* $m = (m_1, m_2, \ldots, m_n)$ *can be chosen with* $m_n > 0$.

Let $H^+ := \{x \in H \mid [x]_n \geq M_n\}$ be a half-hyperplane in H.

Corollary 3.5 *There is a neighborhood* N_1 *of* M *such that* $W^u \cap H^+ \cap N_1$ *is an one-dimensional curve* $p(s) = M + sm + o(s^2)$ *and*

$$\frac{\partial p(s)}{\partial s} = (b_1(s), b_2(s), \ldots, b_n(s)), \quad b_n(s) > 0. \tag{24}$$

We shall denote $N := N_1 \cap N$, or, in other words, we assume without loss of generality that $N_1 \subset N$. It follows from (24) that we can reparameterize the 1–dimensional curve $p(s)$ using the n-th coordinate $[p(s)]_n - M_n$. The new parameter will be again denoted by s. Let

$$w(s) := W^u \cap H^+ \cap Z_\delta.$$

Let δ_0 be such that for all $\delta \leq \delta_0$ $w(s) \subset p(s)$. It follows from the reparameterization of $p(s)$ that $w(s) = p(s)$ for $s \in [0, \delta]$, $\delta \leq \delta_0$ and, in particular, $w(\delta) \in J_\delta$.

Proof of Lemma 3.4. Since $F(x) = G^*_\delta(x)$ in Z_δ and $\gamma(t) \subset Z_\delta$, the linear variational system around $\gamma(t)$ is a $\mathcal{MCFS}$. This allows us to use the results from section 2.

By Lemma 3.3 $TW^u_\Gamma(0)$ is transverse to H and since $dim\, TW^u_\Gamma = 2$ and $dim\, H = n - 1$ it follows that $TW^u_\Gamma(0) \cap H$ is a line.

Let $q = (q_1, \ldots, q_n)$ be a unit vector spanning this line. If we show that $q_n \neq 0$ then $m = +q$ or $m = -q$ satisfies the conclusion of the Lemma.

Assume $q_n = 0$. Since $TW^u_\Gamma(0)$ is two dimensional, $\dot\gamma(0), q \in TW^u_\Gamma(0)$ and $\dot\gamma(0) \notin H$, $q \in H$, it follows that the vectors $q, \dot\gamma(0)$ form a basis of $TW^u_\Gamma(0)$.

Take constants c_1, c_2 such that

$$[c_1\dot\gamma(0) + c_2 q]_{n-1} = 0.$$

Since $[\dot\gamma(0)]_n = 0$ and from the assumption $q_n = 0$ we have that

$$[c_1\dot\gamma(0) + c_2 q]_n = 0.$$

Now $z = c_1\dot\gamma(0) + c_2 q \in TW^u_\Gamma(0)$ but by Theorem 2.1 $N(z)$ is not defined, because $([z]_n, [z]_{n-1}) = (0, 0)$. However, by Lemma 2.3 with $\Delta = -1$ and $k = 1$ $N(z) = 1$, which is a contradiction. □

We have just proved that the unstable manifold of $\gamma(t)$ sticks out of Z_δ. We need an analogous lemma about the stable manifold W^s of γ. Let α be a unit perpendicular vector to $TW^s_\Gamma(0)$. Since $m \in TW^u_\Gamma(0)$ and m is not a multiple of $\dot\gamma(0)$, we have $\alpha \cdot m \neq 0$.

Let $\delta_1 \leq \delta_0$ be such that for all $\delta \leq \delta_1$ also $\alpha \cdot (w(\delta) - M) \neq 0$. To avoid the ambiguity connected with a direction of α we fix the direction by requiring

$$\alpha \cdot (w(\delta) - M) > 0. \tag{25}$$

Lemma 3.6 $[\alpha]_1 \neq 0$, $[\alpha]_n \neq 0$.

Remark 3.7 The statement $[\alpha]_1 \neq 0$ implies that $TW^s_\Gamma(0)$ is not parallel to J_δ and since they are both hyperplanes, $TW^s_\Gamma(0) \pitchfork J_\delta$. Thus $TW^s_\Gamma(0)$ "sticks out" of $\mathbf{R}^n \setminus Z_\delta$ and so for small δ we have $W^s \cap (\mathbf{R}^n \setminus Z_\delta) \neq \emptyset$. In this respect the statements of Lemma 3.6 and Lemma 3.4 are analogous. The statement $[\alpha_n] \neq 0$ will be used later in the proof.

Proof. We shall again use the results from section 2, since the linear variational equation around $\gamma(t)$ is a $\mathcal{MCFS}$.

Let K_i be a set (in fact an open cone) on which the function N is constant and equal to i. Observe, that K_i is an open set for each i.

Suppose $[\alpha]_1 = 0$. Then the line $y = ce_1 \in TW^s_\Gamma(0)$ where e_1 is the first standard unit vector. In particular, $e_1 \in TW^s_\Gamma(0)$ and so $e_1 = c_1 z + c_2\dot\gamma(0)$, $z \in \pi_s^{-1}(0)$ for some constants c_1, c_2. By Lemma 2.3

$$N(c_1 z) \geq 3. \tag{26}$$

On the other hand

$$c_1 z = e_1 - c_2\dot\gamma(0) \tag{27}$$

and thus $[z]_n = 0$ because $[\dot\gamma(0)]_n = 0$.

If $c_2 > 0$ then by Lemma 2.6 $[c_1 z]_i > 0$ for $i \neq n$ and from the definition of the function $N(\cdot)$ we get $N(c_1 z) = 1$. Similarly, if $c_2 < \frac{1}{[\dot\gamma(0)]_1} < 0$ then from Lemma 2.6 we get $N(c_1 z) = 1$.

If $\frac{1}{[\dot\gamma(0)]_1} \leq c_2 < 0$ then $N(c_1 z)$ is not defined, but observe that $c_1 z$ lies on the boundary between K_1 and K_3.

Finally, if $c_2 = 0$ then $N(c_1 z) = N(e_1)$ is not defined, but since $N((1, \epsilon, \dots, \epsilon)) = 1$ we see that $c_1 z$ lies on the boundary of K_1.

In either case this establishes the contradiction with (26).

Now we assume that $[\alpha]_n = 0$ and try to obtain a contradiction as in the previous case. Observe that under our assumption $y = ce_n \subset TW^u_\Gamma(0)$, where e_n is the n-th standard unit vector, and so there are constants c_1, c_2 such that

$$e_n = c_1 z + c_2\dot\gamma(0)$$

where $z \in \pi_s^{-1}(0)$. Therefore $c_1 z = e_n - c_2\dot\gamma(0)$ and $N(c_1 z) \geq 3$.

Now if $c_2 > 0$ or $c_2 < 0$ then $N(c_1 z) = 1$ and if $c_2 = 0$ we get as above that though $N(c_1 z)$ is not defined, $c_1 z = e_n$ lies on the boundary of K_1.

In either case we arrive to contradiction with (26). □

Remark 3.8 We used Lemma 2.6 and thus the discrete Ljapunov function in an essential way in the proofs of Lemma 3.4 and Lemma 3.6. Any attempt to generalize our results beyond the structure of $\mathcal{MCFS}$ will have to take this into account.

3.3 Step 2: Flow defined maps.

We now introduce flow defined maps $\varphi, \hat{\varphi}_\delta, \check{\varphi}_\delta, \psi_\delta, \hat{\psi}_\delta$ and $\check{\psi}_\delta$, which will help us to describe the differences between the flows Φ, $\check{\Phi}_\delta$ and $\hat{\Phi}_\delta$. While the first three maps describe the flows between J_δ^+ and H, the other three between H and J_δ^-. As we shall see, there are similarities between the two sets of maps.

We will use the notation ψ_δ^* for $\hat{\psi}_\delta$ and $\check{\psi}_\delta$ and the notation φ_δ^* for $\hat{\varphi}_\delta$ and $\check{\varphi}_\delta$.
Let $\varphi_\delta : J_\delta^+ \to H$ be a map defined by

$$\varphi_\delta(x) = \Phi(x, T_\delta(x))$$

where $T_\delta(x) = min\{t > 0 \mid \Phi(x,t) \in H\}$ and let $\varphi_\delta^* : J_\delta^+ \to H$ be defined by

$$\varphi_\delta^*(x) = \Phi_\delta^*(x, T_\delta^*(x)).$$

where $T_\delta^*(x)$ is the analog of $T_\delta(x)$ for the flow Φ_δ^*. Observe that the last definition is in fact a definition of two maps, $\hat{\varphi}_\delta$ and $\check{\varphi}_\delta$, and two times $\hat{T}_\delta(x)$ and $\check{T}_\delta(x)$, defined by the flows $\hat{\Phi}_\delta$ and $\check{\Phi}_\delta$ respectively.

Now we define maps ψ. Let $\psi_\delta : H \to J_\delta^-$ be a map defined by

$$\psi_\delta(x) = \Phi(x, \tau_\delta(x))$$

where $\tau_\delta(x) = min\{t > 0 \mid \Phi(x,t) \in J_\delta^-\}$ and let $\psi_\delta^* : H \to J_\delta^-$ be defined by

$$\psi_\delta^*(x) = \Phi_\delta^*(x, \tau_\delta^*(x)).$$

where $\tau_\delta^*(x)$ is the analog of $\tau_\delta(x)$ for the flow Φ_δ^*. The last definition is again a definition of two maps, $\hat{\psi}_\delta$ and $\check{\psi}_\delta$, and two times $\hat{\tau}_\delta(x)$ and $\check{\tau}_\delta(x)$ defined by the flows $\hat{\Phi}_\delta$ and $\check{\Phi}_\delta$ respectively.

Let $B(x, \epsilon)$ denotes a closed n-dimensional ball centered at x with the radius ϵ. Observe, that since $N \subset H$ is a Poincarè section for Φ and Φ_δ^* for all δ and $[F(x)]_1 - [G_\delta^*(x)]_1$ is uniformly bounded as $\delta \to 0$ in an bounded neighborhood of M, there is a ball $B(M, \mu)$, independent of δ, such that if $x \in \mathcal{G}^+ := B(M,\mu) \cap J_\delta^+$ then $\varphi_\delta(x)$ and $\varphi_\delta^*(x)$ are defined. Let $\mathcal{G}^- := B(M,\mu) \cap J_\delta^-$.

We want to point out that though the sets $\mathcal{G}^+$ and $\mathcal{G}^-$ change with δ, they do not shrink to a point as $\delta \to 0$, since μ in $B(M,\mu)$ is independent on δ. In other words, the size of $\mathcal{G}^+$ and $\mathcal{G}^-$ is bounded away from zero as $\delta \to 0$.

Since $N \subset H$ is a Poincarè section there is also a neighborhood $B(M, \nu)$ such that for each $x \in H \cap B(M,\nu)$ the maps ψ_δ and ψ_δ^* are defined and ν does not depend on δ.

We shall also need a map

$$\bar{\varphi}_\delta : J_\delta^+ \to H$$

given by the linearized flow $d\Phi(x, 0, \underline{t}(x))$ along γ where $\underline{t}$ is the minimal time such that $d\Phi(x, 0, \underline{t}(x)) \in H$.

For $x \in N \cap J_\delta$ we extend the functions $T_\delta(x), T_\delta^*(x), \tau_\delta(x)$ and $\tau_\delta^*(x)$ by continuity to

$$T_\delta(x) = T_\delta^*(x) = \tau_\delta(x) = \tau_\delta^*(x) = 0.$$

3.4 Step 3: families and p-families.

The aim of this subsection is to define for every family x_δ an important quantity, by which we shall distinguish different (for our purposes) families.

For $y \in \mathbf{R}^n \setminus Z_\delta$ let

$$d(y) := [y]_n - M_n - \delta$$

be the height above J_δ, measured along the n-th coordinate.

We shall give special names to some continuous families for which the function $d(\cdot)$, evaluated on the family, have nice properties. A continuous function

$$y : [0, \delta'] \to N \cap (\mathbf{R}^n \setminus Z_\delta), \qquad \delta \mapsto y_\delta$$

for some $\delta' > 0$, is called a *p-family* if

$$d(y_\delta) = O(\delta^p) \text{ as } \delta \to 0.$$

A continuous function $x : [0, \delta'] \to \mathcal{G}^+ \subset J_\delta^+$, $\delta \mapsto x_\delta$ such that

$$\Phi(x_\delta, T_\delta(x_\delta)) = \varphi_\delta(x_\delta) \in N \cap (\mathbf{R}^n \setminus Z_\delta)$$

is a p-family, is called a *(p,+)-family.*

3.5 Step 4: Sets S_δ and U_δ.

As we have seen in the step 1, the fact that the linear variational equation along $\gamma(t)$ is a $\mathcal{MCFS}$, gives us certain information about the location of the stable and unstable manifolds of γ under Φ. We want to identify subsets of the stable and unstable manifolds under Φ, which are also subsets of the stable and unstable manifolds under Φ_δ^*. We will use in an essential way the assumption that γ is hyperbolic. In the second part of the following Lemma we formulate a criterion, which can be used to decide whether a point $x \in W^s \cap J_\delta^-$ belongs to S_δ.

Lemma 3.9 *1. There exists $\delta_2 \leq \delta_1$ such that for all $\delta \leq \delta_2$ there is*

- *a set $S_\delta \subset W^s \cap J_\delta^-$ such that $y \in S_\delta$ implies $y \in W_\delta^{s*}$*
- *a set $U_\delta \subset W^u \cap J_\delta^+$ such that $y \in U_\delta$ implies $y \in W_\delta^{u*}$.*

2. Assume that $x_\delta \in W^s \cap J_\delta^-$ is such that $x_\delta = \psi_\delta^*(y_\delta)$ where $y_\delta \in H$.

For every $\delta \leq \delta_2$ there is $\rho^s(\delta)$ with

$$\rho^s(\delta) - \rho^s = O(\delta^2) \text{ as } \delta \to 0$$

such that the following implication holds: if

$$||y_\delta - M|| \leq \rho^s(\delta)\delta$$

then $x_\delta = \psi_\delta^*(y_\delta) \in S_\delta$.

Proof. By definition of ρ^s, an n-2 dimensional ball $B(M, \rho^s\delta) \subset TW^s_\Gamma(0) \cap H$ has the property, that $d\Pi(B(M, \rho^s\delta)) \subset B(M, \delta) \cap TW^s_\Gamma(0) \cap H$ and, furthermore, $d\Pi^n(B(M, \rho^s\delta)) \in Z_\delta$ for all n. Let

$$S^H_\delta := \{x \in W^s \cap H \mid \Pi^n(x) \in Z_\delta \text{ for all } n\}.$$

Since $W^s \cap H$ is tangent to $TW^s_\Gamma(0) \cap H$ at M, by continuity there is $\delta' \leq \delta_1$ such that for all $\delta \leq \delta'$ there is $\rho^s(\delta)$ with $\rho^s(\delta) - \rho^s = O(\delta^2)$ as $\delta \to 0$ such that

$$\{x \in W^s \cap H \mid ||x - M|| \leq \rho^s(\delta)\delta\} \subset S^H_\delta.$$

By Lemma 3.6 $TW^s_\Gamma(0)$ is not parallel to J_δ and since W^s is tangent to $TW^s_\Gamma(0)$ at M there exists $\delta_2 \leq \delta'$ such that for all $\delta \leq \delta_2$ we have $S^H_\delta \neq \emptyset$.

We define

$$S_\delta := \psi_\delta(S^H_\delta)$$

which means that we flow S^H_δ forward to J^-_δ and get S_δ. This proves statements 1a and 2.

The statement 1b is proved in an analogous way, using the maps Π^{-1} and $d\Pi^{-1}$ instead of Π and $d\Pi$ and the Lemma 3.4 instead of the Lemma 3.6.

□

Remark 3.10 Observe, that

$$S_\delta \subset B(M, k) \tag{28}$$

for some k with $k \to 0$ as $\delta \to 0$. A similar remark applies to the set U_δ.

Since the size of the sets $\mathcal{G}^+, \mathcal{G}^-$ is bounded from below as $\delta \to 0$ there exists $\delta_3 \leq \delta_2$ such that for all $\delta \leq \delta_3$

$$U_\delta \subset \mathcal{G}^+ \qquad \text{and} \qquad S_\delta \subset \mathcal{G}^-.$$

Having defined the set U_δ we determine the range of possible p for a $(p,+)$-family $x_\delta \in U_\delta$. We assume that $\delta \leq \delta_3$ and so $U_\delta \subset \mathcal{G}^+$.

Observe that if $x \in U_\delta \cap H$ then $d(x) = 0$ and if $x_\delta \in U_\delta$ is a family with $\Pi^{-1}(\varphi_\delta(x_\delta)) = w(\delta)$ then

$$\begin{aligned} d(\varphi_\delta(x_\delta)) &= [\varphi_\delta(x_\delta)]_n - M_n - \delta \\ &= [\Pi(\Pi^{-1}(\varphi_\delta(x_\delta)))]_n - M_n - \delta \\ &= [\Pi(w(\delta))]_n - M_n - \delta \\ &= \rho^u(\delta)\delta + M_n - M_n - \delta \\ &= (\rho^u(\delta) - 1)\delta \end{aligned}$$

where $\rho^u(\delta) > 1$ and $\rho^u(\delta) \to \rho^u$ as $\delta \to 0$.

We see that the image of the map $d(\varphi_\delta(\cdot))$ is the interval $[0, (\rho^u(\delta) - 1)\delta]$.

Let $E_\delta : [0, (\rho^u(\delta) - 1)\delta] \to [1, \infty]$ be a map defined by

$$E_\delta(y) = \ln_\delta \frac{y}{\rho^u(\delta) - 1}$$

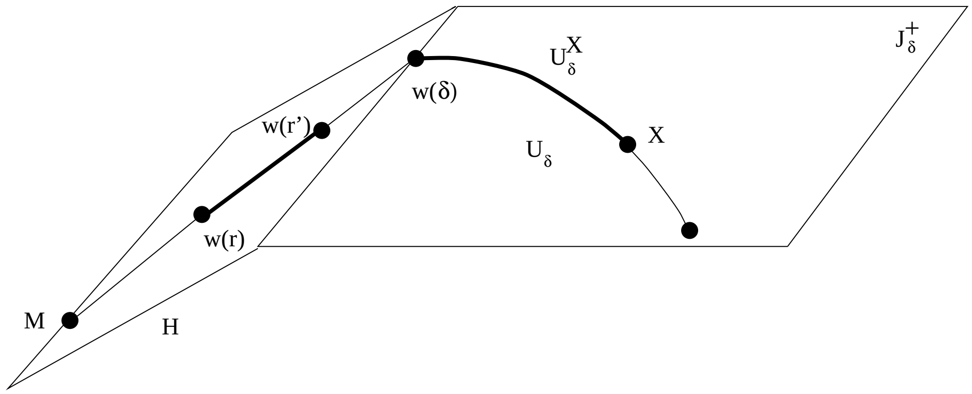


Figure 12: The set $U_\delta^z \subset U_\delta$.

where $\ln_\delta$ is the logarithm with the base δ. We assume without loss that $\delta < 1$ and set $E_\delta(0) = \infty$. Observe that E_δ is a one-to-one monotone function.

Let $q_\delta : U_\delta \to [1, \infty]$ be defined by $q_\delta = E_\delta \circ d \circ \varphi_\delta$. The map $q_\delta(x)$ assigns to every $(p, +)$-family $x_\delta \in U_\delta$ the exponent p of $d(\varphi(x_\delta)) = (\rho^u(\delta) - 1)\delta^p$.

Note that the possible set of values p is the interval $[1, \infty]$.

3.6 Step 5: Estimates of the differences $\varphi_\delta(x_\delta) - \varphi_\delta^*(x_\delta)$ and $\psi_\delta(y_\delta) - \psi_\delta^*(y_\delta)$.

We start the subsection with a definition. Observe that the set U_δ can be written as $U_\delta = \varphi_\delta^{-1}(\Pi(w(s)))$ for $s \in [r, \delta]$ where $r = r(\delta)$ is given by

$$\Pi(w(r)) = w(\delta).$$

Recall that $w(s) = W^u \cap H^+ \cap Z_\delta$ is a segment of a one-dimensional curve. Given any family $z_\delta \in U_\delta$ there is $r'(\delta)$ such that $r'(\delta) \in [r, \delta]$ and $z_\delta = \varphi_\delta^{-1}(\Pi(w(r'))$ for every δ. We let

$$U_\delta^z := \bigcup_{l \in [r, r']} \varphi_\delta^{-1}(\Pi(w(l))).$$

Roughly speaking U_δ^z is a part of U_δ consisting of the points x, such that the trajectory under Φ, starting from x, lie closer (along W^u) to γ then the trajectory starting from z_δ. Since all such trajectories lie in a two dimensional manifold "being closer" is well defined (Figure 12).

Now we shall investigate the difference $\varphi_\delta(x_\delta) - \varphi_\delta^*(x_\delta)$ as $\delta \to 0$ for a family $x_\delta \in U_\delta$ and the difference $\psi_\delta(y_\delta) - \psi_\delta^*(y_\delta)$ for a family $y_\delta \in H$.

It is intuitively clear that the longer the trajectory stays out of Z_δ, the more the trajectory under Φ_δ^* will differ from the one under Φ_δ. This in turn is determined by the time it takes for the flow to get from $x_\delta \in U_\delta$ to $\varphi_\delta(x_\delta) \in H$ and $\varphi_\delta^*(x_\delta) \in H$. This time is given by $T_\delta(x_\delta)$ and $T_\delta^*(x_\delta)$ respectively.

Similarly, the time it takes the flow to get from $y_\delta \in H$ to $\psi_\delta(y_\delta) \in J_\delta^-$ and $\psi_\delta^*(y_\delta) \in J_\delta^-$ is given by $\tau_\delta(y_\delta)$ and $\tau_\delta^*(y_\delta)$ respectively.

The families $x_\delta \in U_\delta$ and $y_\delta \in H$ will be characterized by the function $d(\cdot)$. We assume that for the family x_δ we have $d(\varphi_\delta(x_\delta)) = h(\delta)$ as $\delta \to 0$ for some function $h(\delta)$ and that for the family y_δ we have $d(y_\delta) = i(\delta)$ for some function $i(\delta)$.

We will first estimate the dependence of $T_\delta(x_\delta), T^*_\delta(x_\delta)$ on $h(\delta)$ and the dependence of $\tau(y_\delta)$ and $\tau^*_\delta(y_\delta)$ on $i(\delta)$.

We retained this level of generality since, as outlined in step 6, we will need to consider a family $u^*_\delta \in \varphi^*_\delta(U_\delta) \cap W^s \cap H$ and since it is given very implicitly, we do not know whether u^*_δ is a continuous p-family for some p. With this exception, we shall always use the following Lemmas in the situation where we consider a continuous p-family and $h(\delta) = c\delta^p$ for some constant c.

The estimates are based on the fact, that $[\ddot{\gamma}(0)]_n < 0$ (Lemma 2.5) and that we can approximate the flow in the neighborhood of M by the path of a projectile with initial point x_δ and initial velocity $F(x_\delta) = G^*_\delta(x_\delta)$. The role of gravitational constant g will be played by $[\ddot{\gamma}(0)]_n$. The details are technical in nature and in order not to disrupt the argument, we will postpone the proof of Lemma 3.11 to Section 4.

Lemma 3.11 *1. If $x_\delta \in U_\delta$ is a family with $d(\varphi_\delta(x_\delta)) = h(\delta)$ then*

$$T_\delta(x_\delta) = O(h(\delta)^{\frac{1}{2}}) \qquad T^*_\delta(x_\delta) = T_\delta(x_\delta) + o(T^*_\delta(x_\delta)) \text{ as } \delta \to 0.$$

Furthermore, there is $\delta_4 \le \delta_3$ such that for all $\delta \le \delta_4$, if $z_\delta \in U^x_\delta$ is a family, then

(a)

$$T_\delta(z_\delta) \le T_\delta(x_\delta) \tag{29}$$

(b)

$$d(\varphi_\delta(z_\delta)) \le d(\varphi_\delta(x_\delta)) \tag{30}$$

(c)

$$d(\varphi^*_\delta(z_\delta)) \le O(d(\varphi^*_\delta(x_\delta))) \tag{31}$$

as $\delta \to 0$.

2. If $y_\delta \in H$ is a family with $d(y_\delta) = \iota(\delta)$ then

$$\tau_\delta(y_\delta) = O(\iota(\delta)^{\frac{1}{2}}) \qquad \tau^*_\delta(y_\delta) = \tau_\delta(y_\delta) + o(\tau^*_\delta(y_\delta)) \text{ as } \delta \to 0.$$

We can turn the above estimates on time into the estimates of $\varphi_\delta(x_\delta) - \varphi^*_\delta(x_\delta)$ and $\psi_\delta(y_\delta) - \psi^*_\delta(y_\delta)$. Lemma 3.12 is the crucial step in the proof of Theorem 3.1. The proof can be found in Section 4.

Lemma 3.12 *1. (a) Let $x_\delta \in U_\delta$ be a family with $d(\varphi_\delta(x_\delta)) = h(\delta)$. If $\eta(\delta) = \delta^k$ with k such that*

$$\limsup_{\delta\to 0} \frac{\delta^k}{h(\delta)} = 0$$

then there is $\delta_5 \le \delta_4$ such that for all $\delta \le \delta_5$

$$\varphi_\delta(x_\delta) - \varphi^*_\delta(x_\delta) = c^*(x_\delta)e_1 + O(h(\delta))$$

where e_1 is the first unit vector in $\mathbf{R}^n$ and

$$|c^*(x_\delta)| = O(h(\delta)^{\frac{1}{2}}).$$

Furthermore, for all $\delta \le \delta_5$ we have $\hat{c}(x_\delta) < 0$ and $\check{c}(x_\delta) > 0$ i.e. the change is in the opposite direction under $\hat{\Phi}_\delta$ and $\check{\Phi}_\delta$.

(b) If $q_\delta \in U_\delta^x$ is a family, then

$$||\varphi_\delta(q_\delta) - \varphi_\delta^*(q_\delta)|| \le ||\varphi_\delta(x_\delta) - \varphi_\delta^*(x_\delta)|| + O(h(\delta)) \text{ as } \delta \to 0.$$

2. *Let $y_\delta \in H$ be a family with $d(y_\delta) = \iota(\delta)$. If $\eta(\delta) = \delta^k$ with k such that*

$$\limsup_{\delta\to 0} \frac{\delta^k}{\iota(\delta)} = 0$$

then for all $\delta \le \delta_5$

$$\psi_\delta(y_\delta) - \psi_\delta^*(y_\delta) = C^*(y_\delta)e_1 + O(\iota(\delta))$$

where

$$|C^*(y_\delta)| = O(\iota(\delta)^{\frac{1}{2}}).$$

Furthermore, for $\delta \le \delta_5$ we have $\hat{C}(y_\delta) < 0$ and $\check{C}(y_\delta) > 0$.

3.7 Step 6: Topological crossings.

Let us consider a $(\frac{3}{2}, +)$-family $X_\delta \in U_\delta$ and let us set $\eta := \delta^k$ for some $k > 2$. In this case $\varphi_\delta(X_\delta) = h(\delta) = O(\delta^{\frac{3}{2}})$ as $\delta \to 0$ and so

$$\limsup_{\delta \to 0} \frac{\delta^k}{h(\delta)} = \lim_{\delta \to 0} \frac{\delta^k}{h(\delta)} = 0.$$

Hence the Lemma 3.12 applies. Observe, that the choice of $\eta = \eta(\delta)$ implies the properties of the family $\mathcal{F}$ stated in Theorem 3.1.

As alluded to in the outline we first show that $\varphi_\delta^*(U_\delta^X)$ intersect topologically $W^s \cap H$. The first step is to show that the point $\varphi_\delta^*(X_\delta)$ is on the other side of $TW_\Gamma^s(0)$ then the point $w(\delta) = \varphi_\delta^*(w(\delta))$. To do that we establish, that for sufficiently small δ

$$\alpha \cdot (\varphi_\delta^*(X_\delta) - M) < 0, \tag{32}$$

$$|\alpha \cdot (\varphi_\delta^*(X_\delta) - M)| = O(\delta^{\frac{3}{4}}) \text{ as } \delta \to 0. \tag{33}$$

We start by estimating (32) using Lemma 3.12 with $h(\delta) = O(\delta^{\frac{3}{2}})$:

$$\begin{aligned} \alpha \cdot (\varphi_\delta^*(X_\delta) - M) &= \alpha \cdot (\varphi_\delta(X_\delta) - c^*(X_\delta)e_1 - M + O(\delta^{\frac{3}{2}})) \\ &= \alpha \cdot (\varphi_\delta(X_\delta) - M) - [\alpha]_1 c^*(X_\delta) + O(\delta^{\frac{3}{2}})). \end{aligned} \tag{34}$$

Lemma 3.13 *$\alpha \cdot (\varphi_\delta(X_\delta) - M) = O(\delta)$ as $\delta \to 0$.*

Proof. Recall that $\bar{\varphi}_\delta : J_\delta^+ \to H$ is the map defined by the linear flow around γ. It follows that for sufficiently small δ,

$$||\varphi_\delta(X_\delta) - M|| = ||\bar{\varphi}_\delta(X_\delta) - M|| + O(||\bar{\varphi}_\delta(X_\delta) - M||^2).$$

By the cosine rule

$$\alpha \cdot (\varphi_\delta(X_\delta) - M) = ||\alpha||||\varphi_\delta(X_\delta) - M|| \cos\beta(\delta) \tag{35}$$

where $\beta(\delta)$ is the angle between α and $\varphi_\delta(X_\delta) - M$. Since $W^u \cap H$ is tangent to $TW^u_\Gamma(0) \cap H$

$$\cos\beta(\delta) = \alpha \cdot m + O(\delta^2).$$

We observe that for any $z_\delta \in U_\delta$

$$\delta \leq ||\bar{\varphi}_\delta(z_\delta) - M|| \leq \rho^u(\delta)\delta$$

where $\rho^u(\delta) \to \rho^u$ as $\delta \to 0$. Therefore, for any $z_\delta \in U_\delta$

$$||\bar{\varphi}_\delta(z_\delta) - M|| = O(\delta) \text{ as } \delta \to 0. \tag{36}$$

Since $||\varphi_\delta(X_\delta) - M|| \cos\iota(\delta) = ||\bar{\varphi}_\delta(X_\delta) - M||(\alpha \cdot m) + O(\delta^2)$ the result follows from (35), (36) and the fact that $||\alpha|| = 1$. □

Since X_δ is $(\frac{3}{2}, +)$-family and so $h(\delta) = O(\delta^{\frac{3}{2}})$ as $\delta \to 0$, it follows from Lemma 3.12 that

$$|c^*(X_\delta)| = O(\delta^{\frac{3}{4}}) \text{ as } \delta \to 0. \tag{37}$$

In order to prove (32) using (34) we need to have

$$[\alpha]_1 c^*(X_\delta) > 0.$$

This is the place where the difference between the flows $\hat{\Phi}_\delta$ and $\check{\Phi}_\delta$ comes into the play. By Lemma 3.12 for all $\delta \leq \delta_5$

$$\check{c}(X_\delta) > 0 \qquad \text{and} \qquad \hat{c}(X_\delta) < 0.$$

Since by Lemma 3.6 $[\alpha]_1 \neq 0$, it follows that

$$[\alpha]_1 \hat{c}(X_\delta) > 0 \text{ if } [\alpha]_1 < 0 \tag{38}$$

$$[\alpha]_1 \check{c}(X_\delta) > 0 \text{ if } [\alpha]_1 > 0. \tag{39}$$

Now using (37) and Lemma 3.13 we obtain the desired estimate from (34)

$$\alpha \cdot (\varphi^*_\delta(X_\delta) - M) < 0$$

$$|\alpha \cdot (\varphi^*_\delta(X_\delta) - M)| = O(\delta^{\frac{3}{4}}) \text{ as } \delta \to 0$$

for all $\delta \leq \delta_5$, where by (38) and (39),

$$\text{if} \quad [\alpha]_1 < 0 \quad \text{then } \varphi^*_\delta \text{ stands for } \hat{\varphi}_\delta \tag{40}$$

$$\text{and if } [\alpha]_1 > 0 \text{ then } \varphi^*_\delta \text{ stands for } \check{\varphi}_\delta. \tag{41}$$

In what follows we shall keep the notation φ^*_δ. However, one should keep in mind that all the constructions, which we shall do using φ^*_δ, are done **either** for $\hat{\varphi}_\delta$ **or** for $\check{\varphi}_\delta$, depending on the sign of $[\alpha]_1$. Using the argument mentioned in the outline we will rule out the "or" part at the end of the proof.

By the definition of α

$$\alpha \cdot (w(\delta) - M) > 0 \tag{42}$$

and by Lemma 3.6 $|[\alpha]_n| > 0$. So

$$|\alpha \cdot (w(\delta) - M)| \geq |[\alpha]_n \cdot ([w(\delta)]_n - M_n)| = |[\alpha]_n|\delta = O(\delta) \tag{43}$$

as $\delta \to 0$.

A comparison of (32) and (42) shows that the points $\varphi_\delta^*(X_\delta)$ and $w(\delta)$ lie on the opposite sides of $TW_\Gamma^s(0) \cap H$.

Our goal now is to construct neighborhoods $Q_\delta \subset H^+$ and $V_\delta \subset Q_\delta$ which satisfy Theorem 2.9 with $B_\delta \subset Q_\delta \cap W^s$ and $A_\delta \subset \varphi_\delta^*(U_\delta)$ for small δ. The set V_δ will be a neighborhood of the set $TW_\Gamma^s(0) \cap Q_\delta$, such that $B_\delta \subset V_\delta$. In the construction of V_δ we will use the fact that W^s is tangent to $TW_\Gamma^s(0)$ at M. This is the reason for the following construction. Let $m(\mathbf{x}) : \mathbf{R}^{n-1} \to \mathbf{R}$ be a C^1 function with $m(0) = 0$ and

$$m(\mathbf{x}) = O(||\mathbf{x}||^2) \text{ for } ||\mathbf{x}|| \leq b. \tag{44}$$

Let us consider the graph of the function $m(\mathbf{x})$ as a subset of $\mathbf{R}^{n-1} \times \mathbf{R} = \mathbf{R}^n$. Let $i : \mathbf{R}^n \to \mathbf{R}^n$ be an isometry which satisfy

$$i(\mathbf{R}^{n-1} \times 0) = TW_\Gamma^s(0), \qquad i(\mathbf{0}) = M.$$

Since W^s is tangent to $TW_\Gamma^s(0)$ at M, there exists a function $m(x)$ with the above properties such that $i(x, m(x)) \subset W^s$ for all $x \in \mathcal{M}$, where

$$\mathcal{M} = \{x \in TW_\Gamma^s(0) \mid ||x|| \leq b\}.$$

Observe that $\mathcal{M}$ is a neighborhood of M in $TW_\Gamma^s(0)$ and it does not change with δ.

We define

$$A_\delta := \varphi_\delta^*(U_\delta^X).$$

Observe, that A_δ is an one-dimensional manifold with two endpoints $\varphi_\delta^*(X_\delta)$ and $\varphi_\delta^*(w(\delta)) = w(\delta)$.

We will construct a neighborhood $Q_\delta \subset H^+$ of the form

$$Q_\delta := (B(M, \kappa(\delta)) \times [-v(\delta)\alpha, v(\delta)\alpha]) \cap H^+$$

where $B(M, \kappa(\delta)) \subset TW_\Gamma^s(0)$ is a n-2 dimensional ball centered at M with radius $\kappa(\delta)$, α is the perpendicular unit vector to $TW_\Gamma^s(0)$ and $v(\delta)$ is the "thickness" of Q_δ in the perpendicular direction to $TW_\Gamma^s(0)$. We let

$$V_\delta := (B(M, \kappa(\delta)) \times [-\zeta(\delta)\alpha, \zeta(\delta)\alpha]) \cap H^+.$$

Now we determine the functions $\kappa(\delta)$, $v(\delta)$ and $\zeta(\delta)$ so that the Theorem 2.9 can be applied.

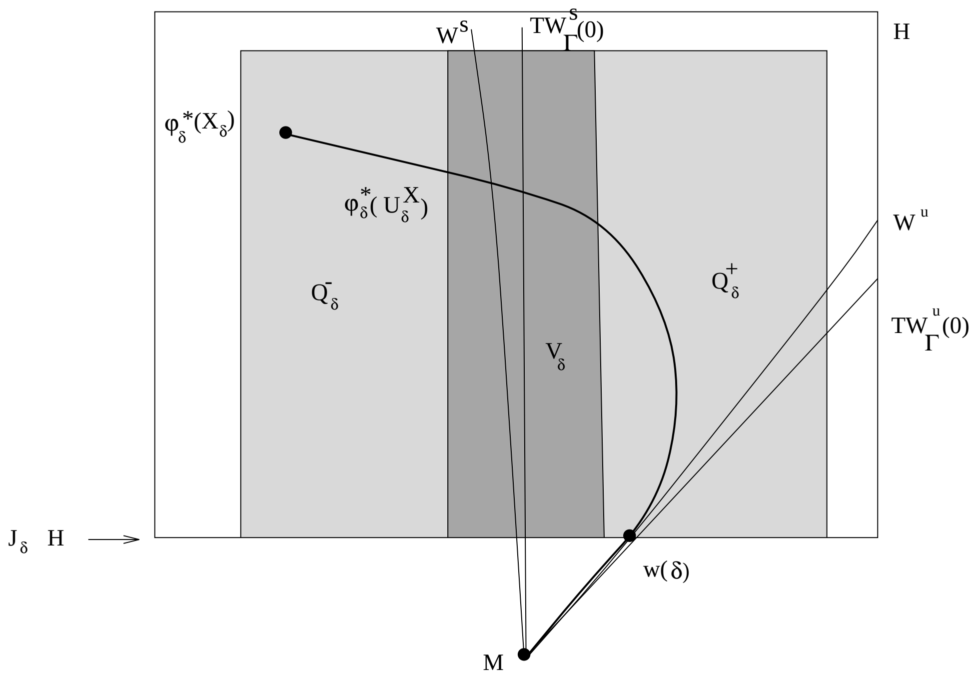


Figure 13: Sets $Q_\delta \subset H$ and $V_\delta \subset Q_\delta$.

Lemma 3.14 *There are functions $\kappa(\delta)$, $v(\delta)$, $\zeta(\delta)$ and $\delta_6 \leq \delta_5$ such that for all $\delta \leq \delta_6$*

1. $W^s \cap Q_\delta \subset V_\delta$
2. W^s *divides* Q_δ *into two parts* Q_δ^+ *and* Q_δ^-
3. *if we let*
$$\mathcal{Q}_\delta^+ := Q_\delta^+ \setminus (Q_\delta \cap V_\delta) \qquad \mathcal{Q}_\delta^- := Q_\delta^- \setminus (Q_\delta \cap V_\delta)$$
then $w(\delta) \in \mathcal{Q}_\delta^+$ *and* $\varphi_\delta^*(X_\delta) \subset \mathcal{Q}_\delta^-$.

Proof. Let $y_\delta \in U_\delta^X$ be a $(q,+)$-family. By (30) $y_\delta \in U_\delta^X$ implies that $q \geq \frac{3}{2}$, since X is a $(\frac{3}{2},+)$-family. By the triangular inequality

$$||\varphi_\delta^*(y_\delta) - M|| \leq ||\varphi_\delta^*(y_\delta) - \varphi_\delta(y_\delta)|| + ||\varphi_\delta(y_\delta) - \bar{\varphi}_\delta(y_\delta)|| + ||\bar{\varphi}_\delta(y_\delta) - M||. \qquad (45)$$

Since $q \geq \frac{3}{2}$ the assumptions of Lemma 3.12.1a are satisfied and

$$||\varphi_\delta^*(y_\delta) - \varphi_\delta(y_\delta)|| = |c^*(y)|e_1 + O(\delta^q) \leq O(\delta^{\frac{3}{4}}) + O(\delta^{\frac{3}{2}}).$$

Since W^u is tangent to $TW_\Gamma^u(0)$ there is $\delta' \leq \delta_5$ such that for all $\delta \leq \delta'$ we have

$$||\varphi_\delta(y_\delta) - \bar{\varphi}_\delta(y_\delta)|| = O(\delta^{2q}).$$

Since by (36) $||\bar{\varphi}_\delta(y_\delta) - M|| = O(\delta)$ as $\delta \to 0$ it follows from (45) and the last two estimates that

$$||\varphi_\delta^*(y_\delta) - M|| \leq O(\delta^{\frac{3}{4}}) \text{ as } \delta \to 0 \qquad (46)$$

Since we can always write U_δ^X as a union of such $(q,+)$-families, (46) implies that $\varphi_\delta^*(U_\delta^*)$ is in $O(\delta^{\frac{3}{4}})$ neighborhood of M. Thus there are functions $\kappa(\delta) = O(\delta^{\frac{3}{4}})$ and $v(\delta) = O(\delta^{\frac{3}{4}})$ as $\delta \to 0$ such that $\varphi_\delta^*(U_\delta^X) \subset Q_\delta$.

Since W^s is tangent to $TW_\Gamma^s(0)$ there is $\delta'' \leq \delta'$ such that for $\delta \leq \delta''$ the projection of $W^s \cap Q_\delta$ onto $TW_\Gamma^s(0) \cap H^+$ is a subset of $\mathcal{M}$. It follows that we can choose the function $\zeta(\delta) = O(\delta^{\frac{3}{2}})$ such that $W^s \cap Q_\delta \subset V_\delta$. This proves the first statement of the Theorem.

Since $TW_\Gamma^s(0) \pitchfork H$ it is easy to see that there is $\delta''' \leq \delta''$ such that for all $\delta \leq \delta'''$ W^s divides Q_δ into two parts.

Finally, since $\zeta(\delta) = O(\delta^{\frac{3}{2}})$ as $\delta \to 0$, it follows from (32),(42),(33),(43) that there is $\delta_6 \leq \delta'''$ such that for all $\delta \leq \delta_6$ the third part of the theorem holds. □

By Theorem 2.9 we can conclude that $\varphi_\delta^*(U_\delta^X)$ and $W^s \cap H$ have topological intersection, which in the view of the dichotomy (40) and (41) means that either $\hat{\varphi}_\delta(U_\delta^X)$ and $W^s \cap H$ have topological intersection or $\check{\varphi}_\delta(U_\delta^X)$ and $W^s \cap H$ have topological intersection. This finishes the first part of the proof of Theorem 3.1, as mentioned in the step 6 of the Outline.

Since we want to show that the sets $\psi_\delta^*(\varphi_\delta^*(U_\delta^X)) \subset W^{u*}$ and $S_\delta \subset W^{s*}$ have topological intersection, the next step is to push the intersection from the hyperplane H to J_δ^-.

Let us denote $\Upsilon_\delta^* = \varphi_\delta^*(U_\delta^X) \cap W^s \cap H$. We have just shown that $\Upsilon_\delta^* \neq \emptyset$ for all $\delta \leq \delta_6$. Let us fix a selector function $\{u_\delta^*\}_{\delta \in (0,\delta_6]} \in \Upsilon_\delta^*$, which means that for every $\delta \leq \delta_6$ there is a unique $u_\delta^* \in \{u_\delta^*\}_{\delta \in (0,\delta_6]}$ such that $u_\delta^* \subset \Upsilon_\delta^*$. We will drop the brackets and use notation u_δ^* for the selector function $\{u_\delta^*\}_{\delta \in (0,\delta_6]}$. We also note that any such a selector function is a family in the sense of definition in Step 3. Let z_δ be a family such that $u_\delta^* = \varphi_\delta^*(z_\delta)$ for $\delta \in (0, \delta_6]$.

Let us consider the set U_δ^z and investigate the image $\mathcal{U}_\delta := \psi_\delta^*(\varphi_\delta^*(U_\delta^z))$. We will show that $\mathcal{U}_\delta$ and $W^s \cap J_\delta^-$ have topological intersection. To prove that we shall closely follow the procedure which we have used to show that $\varphi_\delta^*(U_\delta^X)$ and $W^s \cap H$ have topological intersection. We have to deal with an additional technical difficulty, that instead of a continuous $(\frac{3}{2},+)$-family X as an endpoint of U_δ^X, the endpoint of U_δ^z is characterized by $d(\varphi_\delta(z_\delta)) = l(\delta)$ and the family z_δ need not to be continuous. However, we will have the upper ($\delta^{\frac{3}{2}}$, since $z \in U_\delta^X$, see Lemma 3.11.1c) and lower bound (δ^k for any $k > 2$, see Lemma 3.15 below) on the decay of $l(\delta)$ to 0 as $\delta \to 0$ which gives

$$O(\delta^k) \leq l(\delta) \leq O(\delta^{\frac{3}{2}}) \text{ as } \delta \to 0 \tag{47}$$

for any $k > 2$.

Lemma 3.15 *Let $l(\delta) := d(u_\delta^*)$. Then $\limsup_{\delta \to 0} \frac{\delta^k}{l(\delta)} = 0$ for all $k > 2$.*

The proof can be found in Chapter 4.

Lemma 3.16 *Let u_δ^* be the above family in $\varphi_\delta^*(U_\delta^X) \cap W^s \cap H$ and let $u_\delta^* = \varphi_\delta^*(z_\delta)$. Then*

$$||u_\delta^* - M|| = ||\varphi_\delta^*(z_\delta) - M|| = C\rho^u\delta + O(\delta^{\frac{3}{2}})$$

as $\delta \to 0$, where the constant $C = \sqrt{(1 - \frac{\alpha \cdot m}{[\alpha]_1})^2 + \sum_{i=2}^n m_i^2}$.

Furthermore, if $x_\delta \in U^z_\delta$ *then*

$$||\varphi^*_\delta(x_\delta) - M|| = (C+2)\rho^u\delta + O(\delta^{\frac{3}{2}})$$

as $\delta \to 0$*.*

We postpone the proof to the section 4.

We are ready to show that $\psi^*_\delta(\varphi^*_\delta(U^z_\delta))$ and $W^s \cap J^-_\delta$ have topological intersection. Observe, that in the proof of the fact that $\varphi^*_\delta(U^X_\delta) \cap W^s \cap H \neq \emptyset$ we used three crucial estimates (32), (33) and (46). We shall establish the analogous estimates taking ψ^*_δ instead of φ^*_δ and u^*_δ instead of X_δ.

We first establish an estimate analogous to (46). We start with the estimate of $||\psi_\delta(u^*_\delta) - u^*_\delta||$ as $\delta \to 0$. We approximate the flow by the Taylor expansion at u^*_δ to get

$$\Phi(t, u^*_\delta) = u^*_\delta + F(u^*_\delta)t + O(t^2).$$

Note that $\psi_\delta(u^*_\delta) = \Phi(\tau_\delta(u^*_\delta), u^*_\delta)$ and $\tau_\delta(u^*_\delta) = O(l(\delta)^{\frac{1}{2}})$ by Lemma 3.11. Hence

$$||\psi_\delta(u^*_\delta) - u^*_\delta|| = ||F(u^*_\delta)\tau_\delta(u^*_\delta)|| + O(\tau^2_\delta(u^*_\delta)) = O(l(\delta)^{\frac{1}{2}}) \text{ as } \delta \to 0. \tag{48}$$

Using Lemma 3.15 we see that the assumptions of Lemma 3.12 are satisfied and so we get

$$||\psi_\delta(u^*_\delta) - \psi^*_\delta(u^*_\delta)|| = |C^*(u^*_\delta)| = O(l(\delta)^{\frac{1}{2}}) \text{ as } \delta \to 0.$$

Now, using the last equation , (48) and Lemma 3.16 we have

$$\begin{aligned} ||\psi^*_\delta(u^*_\delta) - M|| &\leq ||\psi^*_\delta(u^*_\delta) - \psi_\delta(u^*_\delta)|| + ||\psi_\delta(u^*_\delta) - u^*_\delta|| + ||u^*_\delta - M|| \\ &\leq O(l(\delta)^{\frac{1}{2}}) + O(l(\delta)^{\frac{1}{2}}) + O(l(\delta)) \\ &= O(l(\delta)^{\frac{1}{2}}) \end{aligned} \tag{49}$$

which is analogous to the estimate (46).

We want to show that there is $\delta_7 \leq \delta_6$ such that for all $\delta \leq \delta_7$ we have

$$\alpha \cdot (\psi^*_\delta(u^*_\delta) - M) < 0 \tag{50}$$

$$||\alpha \cdot (\psi^*_\delta(u^*_\delta) - M)|| = O(l(\delta)^{\frac{1}{2}}) \text{ as } \delta \to 0 \tag{51}$$

in analogy with (32) and (33).

We use the second part of Lemma 3.12 in the following estimate

$$\begin{aligned} \alpha \cdot (\psi^*_\delta(u^*_\delta) - M) &= \alpha \cdot (\psi_\delta(u^*_\delta) - C^*(u^*_\delta)e_1 - M + O(l(\delta)) \\ &= \alpha \cdot (\psi_\delta(u^*_\delta)) - M) - [\alpha]_1 C^*(u^*_\delta) + O(l(\delta)). \end{aligned}$$

Now we show that

$$\alpha \cdot (\psi_\delta(u^*_\delta) - M) = O(l(\delta)) \text{ as } \delta \to 0$$

which correspond to the statement of the Lemma 3.13. Since by the definition of u^*_δ we have $u^*_\delta \in W^s$ and ψ_δ is defined using the flow Φ, by the invariance of W^s under Φ we get that $\psi_\delta(u^*_\delta) \in W^s$. By Lemma 3.16 $||u^*_\delta - M|| = O(\delta)$ as $\delta \to 0$, which, by (47), gives

$$||u^*_\delta - M|| \leq O(l(\delta)^{\frac{1}{2}}) \text{ as } \delta \to 0.$$

Now using (48) we get

$$\begin{aligned} ||\psi_\delta(u^*_\delta) - M|| &\leq ||\psi_\delta(u^*_\delta) - u^*_\delta|| + ||u^*_\delta - M|| \\ &= O(l(\delta)^{\frac{1}{2}}) \text{ as } \delta \to 0. \end{aligned}$$

Since $TW^s_\Gamma(0)$ is tangent to W^s for sufficiently small δ we have

$$\alpha \cdot (\psi_\delta(u^*_\delta) - M) = O(||\psi_\delta(u^*_\delta) - M||^2) = O(l(\delta)) \text{ as } \delta \to 0.$$

Thus we use the same argument that led to (40) and (41) that (50) and (51) hold. We construct sets $Q'_\delta \subset J^-_\delta$ of the form

$$Q'_\delta := (B(M, \kappa'(\delta)) \times [-v'(\delta)\alpha, v'(\delta)\alpha]) \cap J^-_\delta$$

$$V'_\delta := (B(M, \kappa'(\delta)) \times [-\zeta'(\delta)\alpha, \zeta'(\delta)\alpha]) \cap J^-_\delta$$

where $B(M, \kappa'(\delta)) \subset TW^s_\Gamma(0)$ is a n-2 dimensional ball centered at M with radius $\kappa'(\delta)$, α is the perpendicular unit vector to $TW^s_\Gamma(0)$ and $v'(\delta)$ is the "thickness" of Q'_δ in the perpendicular direction to $TW^s_\Gamma(0)$.

Lemma 3.17 *There are functions $\kappa'(\delta)$, $v'(\delta)$, $\zeta'(\delta)$ and $\delta_8 \leq \delta_7$ such that for all $\delta \leq \delta_8$ we have*

1. $W^s \cap Q'_\delta \subset V'_\delta$
2. *W^s divides Q'_δ into two parts Q'^+_δ and Q'^-_δ*
3. *if we let*
$$\mathcal{Q}'^+_\delta := Q'^+_\delta \setminus (Q'_\delta \cap V'_\delta) \qquad \mathcal{Q}'^-_\delta := Q'^-_\delta \setminus (Q'_\delta \cap V'_\delta)$$
then $w(\delta) \in \mathcal{Q}'^+_\delta$ and $\psi^*_\delta(u^*_\delta) \subset \mathcal{Q}'^-_\delta$.

Proof. Proof is analogous to the proof of Lemma 3.14 and is therefore ommited. □

We apply Theorem 2.9 with $Q = Q'_\delta$, $V = V'_\delta$, $A = \psi^*_\delta(\varphi^*_\delta(U^z_\delta))$ and $B = W^s \cap Q'_\delta$ to conclude that $\psi^*_\delta(\varphi^*_\delta(U^z_\delta))$ and $W^s \cap J^-_\delta$ have topological intersection.

We summarize our result in a Corollary.

Corollary 3.18 *If $W^s \cap H \cap \varphi^*_\delta(U^X_\delta) \neq \emptyset$ and $\varphi^*_\delta(z_\delta) \in W^s \cap H \cap \varphi^*_\delta(U^X_\delta)$ then $W^s \cap J^-_\delta$ and $\psi^*_\delta(\varphi^*_\delta(U^z_\delta))$ have topological intersection in J^-_δ.*

The last step is to show that this intersection is actually the intersection of S_δ and $\psi^*_\delta(\varphi^*_\delta(U^z_\delta))$. To do that we need to invoke the assumption (14).

Corollary 3.19 *If*

1. $(C+2)\rho^u < \rho^s$ *or*
2. $n = 3$

then there is $\delta_9 \leq \delta_8$ such that for all $\delta \leq \delta_9$ the sets $\psi^*_\delta(\varphi^*_\delta(U^z_\delta)) \subset W^{u*}$ and $S_\delta \subset W^{s*}$ have topological intersection.

Proof. In view of Corollary 3.18 we need to show that under our assumptions $\psi_\delta^*(\varphi_\delta^*(U_\delta^z)) \cap W^s \cap J_\delta^- \subset S_\delta$.

Let us assume that $(C+2)\rho^u < \rho^s$. There is $\delta_9 \le \delta_8$ such that for all $\delta \le \delta_9$ we have $(C+2)\rho^u < \rho^s(\delta)$ (see Lemma 3.9) and, also, that $(C+2)\rho^u\delta + O(\delta^{\frac{3}{2}}) < \rho^s(\delta)\delta$. Take any family $x_\delta \in U_\delta^X$ such that $\psi_\delta^*(\varphi_\delta^*(x_\delta)) \in \psi_\delta^*(\varphi_\delta^*(U_\delta^z) \cap W^s \cap J_\delta^-$. Then for $\delta \le \delta_9$

$$||\varphi_\delta^*(x_\delta) - M|| \le (C+2)\rho^u\delta + O(\delta^{\frac{3}{2}}) < \rho^s(\delta)\delta.$$

The statement now follows from Lemma 3.9.b.

Now we prove the Corollary under the assumption $n = 3$. In this case $TW_\Gamma^s(0) \cap H$ is a line. Since W^s is tangent to $TW_\Gamma^s(0)$

$$||\varphi_\delta^*(x_\delta) - M|| = ||P\varphi_\delta^*(x_\delta) - M|| + O(||P\varphi_\delta^*(x_\delta) - M||^2)$$

where P is an orthogonal projection onto $TW_\Gamma^s(0) \cap H$. We will show that

$$||P\varphi_\delta^*(x_\delta) - M|| < \rho^s\delta$$

and then we shall use the same argument as we have used under the first assumption. Using the elementary geometry in the plane H we get that

$$||P\varphi_\delta^*(x_\delta) - M|| < \rho^s\delta \quad \text{if and only if } [P\varphi_\delta^*(x_\delta) - M]_3 < \rho^s\delta\cos\beta \tag{52}$$

where β is the angle between $TW_\Gamma^s(0) \cap H$ and e_3. Thus it is enough to show that $[P\varphi_\delta^*(x_\delta) - M]_3 < \rho^s\delta\cos\beta$. We note that by Lemma 3.6 $\beta \ne \frac{\pi}{2}$. Since $\varphi_\delta^*(x_\delta) \not\subset Z_\delta$, we have $[P\varphi_\delta^*(x_\delta) - M]_3 > \delta$ and so $\rho^s\cos\beta > 1$.

Now we compute $[P\varphi_\delta^*(x_\delta) - M]_3$:

$$\begin{aligned} [P\varphi_\delta^*(x_\delta) - M]_3 &= \delta + d(P\varphi_\delta^*(x_\delta) - M) \\ &= \delta + d(\varphi_\delta^*(x_\delta)) + O(\delta^2). \end{aligned}$$

Since $x_\delta \in U_\delta^X$ using (31) we get

$$d(\varphi_\delta^*(x_\delta)) \le O(d(\varphi_\delta^*(X_\delta))) = O(\delta^{\frac{3}{2}}).$$

Hence

$$[P\varphi_\delta^*(x_\delta) - M]_3 = \delta + O(\delta^{\frac{3}{2}}) + O(\delta^2)$$

and since $\rho^s\cos\beta > 1$ there is $\delta' \le \delta_8$ such that for all $\delta \le \delta'$ the second part of (52) holds. Therefore there is $\delta_9 \le \delta'$ such that for all $\delta \le \delta_9$

$$||\varphi_\delta^*(x_\delta) - M|| < \rho^s\delta + O(\delta^2) \le \rho^s(\delta)\delta.$$

The statement now follows from Lemma 3.9 in the same way as under the first assumption.
□

The last step in the proof is to resolve the dichotomy (38) vs. (39). Observe, that we have proved the following

- $[\alpha]_1 < 0$ if and only if S_δ and $\hat{\psi}_\delta(\hat{\varphi}_\delta(U_\delta^X))$ have topological intersection

- $[\alpha]_1 > 0$ if and only if S_δ and $\check{\psi}_\delta(\check{\varphi}_\delta(U_\delta^X))$ have topological intersection.

Observe, that since $[\alpha]_1$ has a definite sign, only one of these possibilities may occur. The reason for introducing the auxiliary function $\check{g}$ is the following argument, which, as was mentioned in point 5 of the outline, will show that the first part of the dichotomy is true.

Observe that $\mathcal{G}_\delta^-$ is mapped diffeomorphically into H^+ by the map ψ_δ^{*-1}. Since this map is a diffeomorphism, $W_\delta^{s*} \cap H$ and $W_\delta^{u*} \cap H$ have topological intersection, which in turn implies that $W^s_{\Pi^*_\delta}$ and $W^u_{\Pi^*_\delta}$ have topological intersection. Here Π^*_δ is the Poincaré map associated with the cross-section H and the flow Φ^*_δ and $W^s_{\Pi^*_\delta}$, $W^u_{\Pi^*_\delta}$ are the stable and unstable manifolds of Π^*_δ respectively. The dichotomy now reads

- **either** $W^s_{\hat{\Pi}_\delta}$ and $W^u_{\hat{\Pi}_\delta}$ have a topological crossing
- **or** $W^s_{\check{\Pi}_\delta}$ and $W^u_{\check{\Pi}_\delta}$ have a topological crossing.

Assume that the second part of a dichotomy holds. Then $\check{W}^u_\delta \cap \check{W}^s_\delta \neq \emptyset$ by Lemma 2.10, which contradicts the Proposition 1.9, because $\check{g}_\delta$ is a monotone function.

This proves Theorem 3.1 with $\bar{\delta} = \delta_9$ and hence conclude the proof of Theorem 1.11. □

3.8 Proof of Theorem 1.12

Recall that $M_n = max_{t\in[0,\bar{T}]}[\gamma(t)]_n$.

Let $\epsilon > 0$ be given. We use the fact that the Theorem 1.11 holds for all families $\mathcal{G}$ and $\mathcal{F}$ which satisfy the assumptions. We construct $\mathcal{G}$ and $\mathcal{F} \subset \mathcal{G}$ such that for $\hat{g}_{\delta,\eta(\delta)} \in \mathcal{F}$ we have

$$\max_{x\in J} |\hat{g}_{\delta,\eta(\delta)}(x) - f(x)| < \epsilon \tag{53}$$

where $J := (M_n, M_n + 2\eta + \delta + \delta^{\frac{1}{2}})$.

First we choose L in such a way that $f(M_n) - L < \frac{\epsilon}{4}$. Let

$$a := max_{x\in[M_n-\epsilon, M_n+\epsilon]} f'(x).$$

Let δ' be such that

$$a(3\delta' + \delta'^{\frac{1}{2}}) \le \frac{\epsilon}{4} \qquad \text{and} \qquad 3\delta' + \delta'^{\frac{1}{2}} \le \epsilon. \tag{54}$$

Since f is monotonically increasing and for the maximum $\hat{g}(y)$ of $\hat{g}_{\delta,\eta(\delta)}(x)$ we have that $\hat{g}(y) < f(y)$, $sup_{x\in J}|\hat{g}_{\delta,\eta(\delta)}(x) - f(x)|$ is achieved in the right endpoint of the interval J. Therefore

$$\begin{aligned} sup_{x\in J}|\hat{g}_{\delta,\eta(\delta)}(x) - f(x)| &= |\hat{g}_{\delta,\eta(\delta)}(M_n + 2\eta + \delta + \delta^{\frac{1}{2}}) - f(M_n + 2\eta + \delta + \delta^{\frac{1}{2}}| \\ &= |(L - \vartheta) - f(M_n + 2\eta + \delta + \delta^{\frac{1}{2}})| \end{aligned}$$

for some $\vartheta > 0$. The quantity ϑ comes from the requirement that $\hat{g}_{\delta,\eta(\delta)}$ must be decreasing on the interval $[M_n + \delta + \eta, \infty)$ and we can make it as small as we wish, not changing δ and η. Let us choose such a family $\mathcal{G}$ that $\vartheta \le \frac{\epsilon}{2}$.

Now we choose a family $\mathcal{F}$ by taking $\eta(\delta) = \delta^k$ for some $k > 2$. In particular we may assume that $\eta < \delta$ and using (54) we have for $\delta \leq \delta'$

$$\begin{aligned} |f(M_n + 2\eta + \delta + \delta^{\frac{1}{2}}) - (L - \vartheta)| &\leq |f(M_n) - L| + a(2\eta + \delta + \delta^{\frac{1}{2}}) + \vartheta \\ \leq \frac{\epsilon}{4} + a(3\delta' + \delta'^{\frac{1}{2}}) + \vartheta &\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \vartheta \leq \epsilon. \end{aligned}$$

Now we apply the Theorem 1.11 to the constructed families $\mathcal{G}$ and $\mathcal{F}$ to conclude that there is $\bar{\delta} \leq \delta'$ such that for all $\delta \leq \bar{\delta}$ Theorem 1.11 holds for $\hat{g}_{\delta,\eta(\delta)} \in \mathcal{F}$.

To every such a function $\hat{g}_{\delta,\eta(\delta)} \in \mathcal{F}$ we construct a C^1 function $h(x)$ (see Figure 14) as follows

$$\begin{aligned} h(x) &= \hat{g}_{\delta,\eta(\delta)}(x) \text{ if } x \in (-\infty, M_n + \delta + \eta + \delta^{\frac{1}{2}}] \\ h(x) &= f(x) \text{ if } x \in (M_n + \delta + 2\eta + \delta^{\frac{1}{2}}, \infty) \\ h(x) & \quad \text{has a unique minimum in } z \in (M_n + \delta + \eta + \delta^{\frac{1}{2}}, M_n + \delta + 2\eta + \delta^{\frac{1}{2}}]. \end{aligned}$$

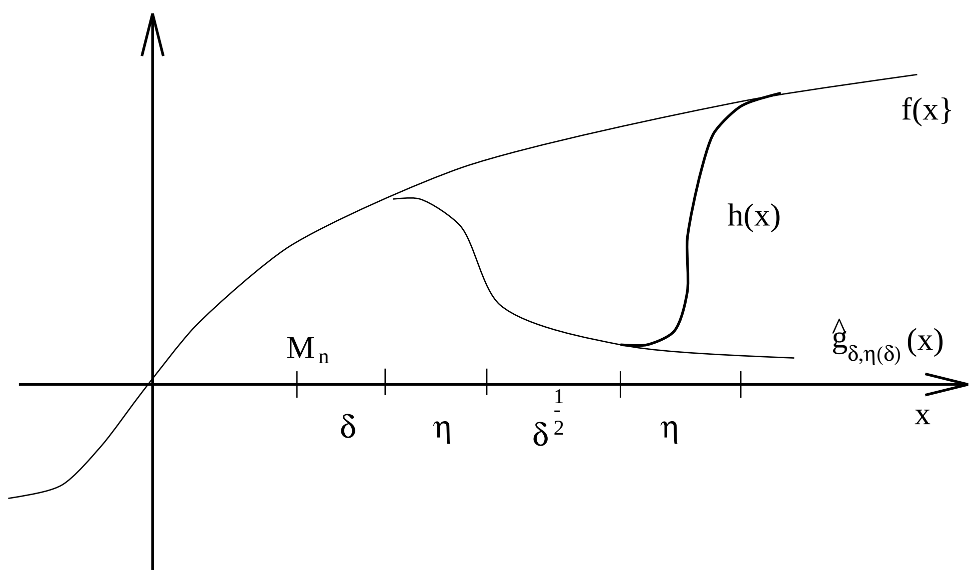


Figure 14: Function $h(x)$.

Observe that by the construction and (53)

$$||h(x) - f(x)||_{C^0} < \epsilon.$$

Now we note that in the proof of Theorem 1.11 we used the properties of the function $\hat{g}_{\delta,\eta(\delta)}(x)$ for $[x]_n \leq M_n + \delta + \eta + O(\delta)$. More precisely, since for all $x \in U_\delta$ we had $d(\varphi_\delta(x)) \leq O(\delta)$ and $d(\varphi^*_\delta(x)) \leq O(\delta)$ (see (46)) as $\delta \to 0$ it follows that

$$d(\varphi_\delta(x)) \leq \delta + \delta^{\frac{1}{2}} \tag{55}$$

for sufficiently small δ for all $x \in U_\delta$. We may assume without loss that (55) holds for $\delta \leq \bar{\delta}$.

Since $\hat{g}_{\delta,\eta(\delta)}(x) = h(x)$ for $x \in (-\infty, M_n + \delta + \delta^{\frac{1}{2}} + \eta]$ all the conclusions of Theorem 1.11 involving $\hat{g}_{\delta,\eta(\delta)} \in \mathcal{F}$ for $\delta \leq \bar{\delta}$ remain valid for the function $h(x)$. □

4 Proofs of the Lemmas.

Before we start the proofs of Lemma 3.11 and Lemma 3.12 we make an estimate which will be used in both proofs.

Let us consider a family $x_\delta \in U_\delta$ and define $Y_\delta(t) := \Phi(x_\delta, t) - \Phi^*_\delta(x_\delta, t)$. Then from the equations (8) we have

$$[\dot{Y}_\delta(t)]_1 = -a_1[Y_\delta(t)]_1 - b_1(f([\Phi(x_\delta, s)]_n) - g^*_\delta([\Phi^*_\delta(x_\delta, s)]_n))$$

and so using the variation of constants formula

$$\begin{aligned} [Y_\delta(t)]_1 &= e^{-a_1 t}[Y_\delta(0)]_1 - b_1 \int_0^t e^{-a_1(t-s)}(f([\Phi(x_\delta, s)]_n) - g^*_\delta([\Phi^*_\delta(x_\delta, s)]_n))ds \\ &= -b_1 \int_0^t e^{-a_1(t-s)}(f([\Phi(x_\delta, s)]_n) - g^*_\delta([\Phi^*_\delta(x_\delta, s)]_n))ds \end{aligned} \tag{56}$$

since $[Y_\delta(0)]_1 = \Phi_\delta(x_\delta, 0) - \Phi^*_\delta(x_\delta, 0) = 0$. For $i = 2, \dots, n$ we compute in a similar way

$$[Y_\delta(t)]_i = b_i \int_0^t e^{-a_i(t-s)}[Y_\delta(s)]_{i-1} ds.$$

Using the last equation inductively, one gets that

$$\frac{d^k}{dt^k}[Y_\delta(t)]_i|_{t=0} = 0 \quad \text{for } k = 0, \dots, i-2$$

$$\frac{d^{i-1}}{dt^{i-1}}[Y_\delta(t)]_i|_{t=0} = [Y_\delta(0)]_1 = 0$$

and, finally,

$$\frac{d^i}{dt^i}[Y_\delta(t)]_i|_{t=0} = [\dot{Y}_\delta(0)]_1 \neq 0.$$

Expanding into a Taylor polynomial at $t = 0$ we get that

$$[Y_\delta(t)]_i = Ct^i + O(t^{i+1}) \tag{57}$$

with $C = [\dot{Y}_\delta(0)]_1$. Observe, that although C depends on δ it is uniformly bounded in a bounded neighborhood of M.

4.1 Proof of Lemma 3.11

Recall that by the assumption $x_\delta \in U_\delta$ is a family with $d(\varphi_\delta(x_\delta)) = h(\delta)$.

We approximate the trajectory $\Phi(x_\delta, t)$, $t \in [0, T_\delta(x_\delta)]$ by a Taylor polynomial centered at the point $x_\delta \in U_\delta$

$$[\Phi(x_\delta, t)]_n = [x_\delta]_n + \frac{d}{dt}[\Phi(x_\delta, t)]_n|_{t=0} t + \frac{1}{2}\frac{d^2}{dt^2}[\Phi(x_\delta, t)]_n|_{t=0} t^2 + O(t^3) \tag{58}$$

Let us denote $v_n(x_\delta) := \frac{d}{dt}[\Phi(x_\delta, t)]_n|_{t=0} = [F(x_\delta)]_n = [G^*_\delta(x_\delta)]_n$ and $a(x_\delta) := \frac{d^2}{dt^2}[\Phi(x_\delta, t)]_n|_{t=0}$.

We approximate $a(x_\delta)$ by $a(M) := [\ddot{\gamma}(0)]_n$ and get

$$\begin{aligned}[\Phi(x_\delta, t)]_n &= [x_\delta]_n + v_n(x_\delta)t + \frac{1}{2}a(M)t^2 + \frac{1}{2}(a(x_\delta) - a(M))t^2 + O(t^3) \\ &= [x_\delta]_n + v_n(x_\delta)t + \frac{1}{2}a(M)t^2 + o(1)O(t^2) + O(t^3) \end{aligned} \tag{59}$$

because $U_\delta \to M$ (Remark 3.10) as $\delta \to 0$ and so $a(x_\delta) \to a(M)$ as $\delta \to 0$.

We observe that first three terms in (59) describe the vertical component of a motion of the projectile in the plane, where $a(M) < 0$ plays the role of a gravitational constant g and $v_n(x_\delta)$ is an initial velocity. A direct calculation shows that the maximal achieved height of the projectile is

$$\begin{aligned}[\Phi(x_\delta, T_\delta(x_\delta))]_n &= M_n + \delta + v_n(x_\delta)(-\frac{v_n(x_\delta)}{a(M)}) + \frac{1}{2}a(M)(-\frac{v_n(x_\delta)}{a(M)})^2 + o(1)O(T_\delta^2(x_\delta)) \\ &= M_n + \delta - \frac{1}{2}\frac{v_n^2(x_\delta)}{a(M)} + o(1)O(T_\delta^2(x_\delta)). \end{aligned} \tag{60}$$

and

$$T_\delta(x_\delta) = -\frac{v_n(x_\delta)}{a(M)} + o(1)O(T_\delta(x_\delta)). \tag{61}$$

From (60) we get

$$d(\varphi_\delta(x_\delta)) = [\Phi(x_\delta, T_\delta(x_\delta))]_n - M_n - \delta = -\frac{v_n^2(x_\delta)}{2a(M)} + o(1)O(T_\delta^2(x_\delta)). \tag{62}$$

Since $d(\varphi_\delta(x_\delta)) = h(\delta)$ comparing (62) and (61) we get

$$\begin{aligned} h(\delta) &= d(\varphi_\delta(x_\delta)) = -\frac{1}{2}T_\delta^2(x_\delta)a(M) + o(T_\delta^2(x_\delta)) \\ &= kT_\delta^2(x_\delta) + o(T_\delta^2(x_\delta)). \end{aligned}$$

Thus

$$T_\delta(x_\delta) = O(h(\delta)^{\frac{1}{2}}) + o(h(\delta)^{\frac{1}{2}}). \tag{63}$$

Let us now estimate

$$\begin{aligned} d(\varphi_\delta^*(x_\delta)) - d(\varphi_\delta(x_\delta)) &= [\Phi_\delta^*(x_\delta, T_\delta^*(x_\delta))]_n - [\Phi(x_\delta, T_\delta(x_\delta))]_n = [\Phi_\delta^*(x_\delta, T_\delta^*(x_\delta))]_n \\ &- [\Phi_\delta^*(x_\delta, T_\delta(x_\delta))]_n + [\Phi_\delta^*(x_\delta, T_\delta(x_\delta))]_n - [\Phi(x_\delta, T_\delta(x_\delta))]_n \\ &= c(T_\delta^*(x_\delta) - T_\delta(x_\delta)) + O(T_\delta^n(x_\delta)) \end{aligned} \tag{64}$$

where the first part holds because $||G_\delta^*(x)||$ is uniformly bounded in a neighborhood of M and the second part follows from (57) with $t = T_\delta(x_\delta)$.

Observe that (58) will be the same if we replace the flow Φ by the flow Φ_δ^* since $x_\delta \in J_\delta$. Hence in analogy with (62)

$$d(\varphi_\delta^*(x_\delta)) = -\frac{v_n^2(x_\delta)}{2a(M)} + o(1)O(T_\delta^*(x_\delta)^2). \tag{65}$$

Thus

$$d(\varphi_\delta^*(x_\delta)) - d(\varphi_\delta(x_\delta)) = o(1)O(T_\delta^*(x_\delta)^2) + o(1)O(T_\delta(x_\delta)^2) \tag{66}$$

and from (64)

$$T_\delta^*(x_\delta) = T_\delta(x_\delta) + o(T_\delta^*(x_\delta)^2) + o(T_\delta(x_\delta)^2) = T_\delta(x_\delta) + o(T_\delta(x_\delta)^2). \tag{67}$$

Now we show that there is $\delta_4 \le \delta_3$ such that for all $\delta \le \delta_4$, if $z_\delta \in U_\delta^x$ then $T_\delta(z_\delta) \le T_\delta(x_\delta)$. Since $TW_\Gamma^u(0) \pitchfork H$ by Lemma 3.4 there is δ' such that for all $\delta \le \delta'$ $U_\delta \pitchfork H$. By Lemma 2.5 $[\ddot\gamma(0)] \ne 0$ and so it follows that there is $\delta_4 \le \delta'$ such that for $\delta \le \delta_4$

$$\frac{d}{dx}(v_n(x))|_{x \in U_\delta} \ne 0$$

and so $v_n(x)$ is a monotone function of $x \in U_\delta$. Now from (61) we have (29).

The statement (30) follows from the Corollary 3.5 for $\delta \le \delta_2$ since both $\varphi_\delta(z_\delta) \in W^u \cap H$ and $\varphi_\delta(x_\delta) \in W^u \cap H$ and d measures the n-th coordinate above J_δ.

Now we prove the statement (31). We use (66) and (30) to get

$$\begin{aligned} d(\varphi_\delta^*(z_\delta)) &= d(\varphi_\delta(z_\delta)) + o(T_\delta^{2*}(z_\delta)) \le d(\varphi_\delta(x_\delta)) + o(T_\delta^{2*}(z_\delta)) \\ &= d(\varphi_\delta^*(x_\delta)) + o(T_\delta^{2*}(z_\delta)) + o(T_\delta^{2*}(x_\delta)) \\ &= d(\varphi_\delta^*(x_\delta)) + o(T_\delta^{2*}(x_\delta)) \end{aligned}$$

where in the last step we used (29). The result follows since $d(\varphi_\delta^*(x_\delta)) = O(T_\delta^{*2}(x_\delta))$ by (65), (66) and (67).

Now we prove the second part of the Lemma. The proof is analogous to the proof of the first part and so we just outline the argument.

We approximate the trajectory $\Phi(y_\delta, t)$, $t \in [0, \tau(x_\delta)]$ by a Taylor polynomial centered at the point $y_\delta \in H$

$$[\Phi(y_\delta, t)]_n = [y_\delta]_n + \frac{d}{dt}[\Phi(y_\delta, t)]_n|_{t=0} t + \frac{1}{2}\frac{d^2}{dt^2}[\Phi(y_\delta, t)]_n|_{t=0} t^2 + O(t^3). \tag{68}$$

Note that since $y_\delta \in H$ we have $\frac{d}{dt}[\Phi(y_\delta, t)]_n|_{t=0} = [F(y_\delta)]_n = [G_\delta^*(y_\delta)]_n = 0$. We let $a(y_\delta) := \frac{d^2}{dt^2}[\Phi(y_\delta, t)]_n|_{t=0}$.

We approximate $a(y_\delta)$ by $a(M) := [\ddot\gamma(0)]_n$ and get in an analogous way as (59)

$$[\Phi(y_\delta, t)]_n = [y_\delta]_n + \frac{1}{2}a(M)t^2 + \frac{1}{2}a(M)t^2 + o(1)O(t^2). \tag{69}$$

In this approximation τ_δ is given by $[\Phi(y_\delta, \tau_\delta)]_n = M_n + \delta$ and $[y_\delta]_n = M_n + \delta + d(y_\delta)$. Thus from (69) we get $d(y_\delta) = -\frac{1}{2}a(M)\tau_\delta^2 + o(\tau_\delta^2)$. By the assumption $d(y_\delta) = \iota(\delta)$ and so

$$\tau_\delta = O(\iota(\delta)^{\frac{1}{2}}) + o(\iota(\delta)^{\frac{1}{2}}) \text{ as } \delta \to 0.$$

Now we observe that (69) holds also for flows Φ_δ^*, since $[F(y_\delta)]_n = [G^*(y_\delta)]_n = 0$. Thus using the same argument $d(y_\delta) = -\frac{1}{2}a(M)\tau_\delta^*(y_\delta) + o(\tau_\delta^{2*}(y_\delta))$ and so

$$\tau_\delta(y_\delta) = \tau_\delta^*(y_\delta) + o(\tau_\delta^{2*}(y_\delta)) \text{ as } \delta \to 0.$$

□

4.2 Proof of Lemma 3.12

Let us consider a family $x_\delta \in U_\delta$ and recall that we have defined $Y_\delta(t) := \Phi(x_\delta, t) - \Phi^*_\delta(x_\delta, t)$ in the calculation leading to (56). Recall that $\eta(\delta) = \delta^k$ with k such that

$$\limsup_{\delta \to 0} \frac{\delta^k}{h(\delta)} = 0$$

where $h(\delta) = d(\varphi_\delta(x_\delta))$. Since $d(\varphi_\delta(x_\delta)) = [\varphi_\delta(x_\delta)]_n - \delta - M_n$, it follows that there exists δ' such that for all $\delta \leq \delta'$ we have $h(\delta) > \eta$ and so for those δ

$$[\Phi(x_\delta, T_\delta(x_\delta))]_n = [\varphi_\delta(x_\delta)]_n > M_n + \delta + \eta.$$

By (66) we can also assume without loss of generality that for $\delta \leq \delta'$

$$[\Phi^*_\delta(x_\delta, T^*_\delta(x_\delta))]_n = [\varphi^*_\delta(x_\delta)]_n > M_n + \delta + \eta.$$

For $\delta \leq \delta'$ we define time $\underline{t}_\delta$ by $[\Phi(x_\delta, \underline{t}_\delta)]_n = M_n + \delta + \eta$ and time $\underline{t}^*_\delta$ by $[\Phi^*(x_\delta, \underline{t}^*_\delta)]_n = M_n + \delta + \eta$. Using a Taylor approximation at x_δ

$$[\Phi(x_\delta, t)]_n = [x_\delta]_n + v_n(x_\delta)t + O(t^2)$$

where $v_n(x_\delta) = [F(x_\delta)]_n = [G^*(x_\delta)]_n$ in notation used in the proof of Lemma 3.11. For $t = \underline{t}_\delta$ and $t = \underline{t}^*_\delta$ this gives

$$\eta = v_n(x_\delta)\underline{t}_\delta + O(\underline{t}_\delta^2) \text{ and } \eta = v_n(x_\delta)\underline{t}^*_\delta + O(\underline{t}_\delta^{2*}) \tag{70}$$

respectively. From (61) and (63) we have

$$|v_n(x_\delta)| = O(h^{\frac{1}{2}}(\delta)) + o(h^{\frac{1}{2}}(\delta)) \text{ as } \delta \to 0,$$

and (70) implies that

$$\underline{t}_\delta = O(\frac{\delta^k}{h^{\frac{1}{2}}(\delta)}) + o(\frac{\delta^k}{h^{\frac{1}{2}}(\delta)}) \quad \text{and} \quad \underline{t}^*_\delta = O(\frac{\delta^k}{h^{\frac{1}{2}}(\delta)}) + o(\frac{\delta^k}{h^{\frac{1}{2}}(\delta)}) \text{ as } \delta \to 0. \tag{71}$$

From the definition of $Y_\delta(t)$ we have

$$\varphi_\delta(x_\delta) - \varphi^*_\delta(x_\delta) = Y_\delta(T_\delta(x_\delta)) + \Phi^*(x_\delta, T_\delta(x_\delta)) - \Phi^*(x_\delta, T^*_\delta(x_\delta)). \tag{72}$$

By Lemma 3.11 $T_\delta(x_\delta) = O(h^{\frac{1}{2}}(\delta))$ and $T_\delta(x_\delta) = T^*_\delta(x_\delta) + o(T^*_\delta(x_\delta))$. Using the Taylor polynomial centered at x_δ in the same way as above we estimate the second component by

$$\begin{aligned} ||\Phi^*(x_\delta, T_\delta(x_\delta)) - \Phi^*(x_\delta, T^*_\delta(x_\delta))|| &= |v_n(x_\delta)||(T_\delta(x_\delta) - T^*_\delta(x_\delta))| + O(T_\delta(x_\delta)^2 + O(T^*_\delta(x_\delta)^2) \\ &= O(h^{\frac{1}{2}}(\delta))o(h^{\frac{1}{2}}(\delta)) \text{ as } \delta \to 0. \end{aligned} \tag{73}$$

Now we estimate the first part in (72). Observe that $Y_\delta(T_\delta(x_\delta))$ is a vector and (57) implies that the first component of the vector dominates the rest of the components. Therefore we shall estimate the first component $[Y_\delta(T_\delta(x_\delta))]_1$ using (56):

$$\begin{aligned} [Y_\delta(T_\delta)]_1 &= -b_1 \int_0^{T_\delta} e^{-a_1(T_\delta - s)}(f([\Phi(x_\delta, s)]_n) - g^*_\delta([\Phi(x_\delta, s)]_n))ds \\ &\quad -b_1 \int_0^{T_\delta} e^{-a_1(T_\delta - s)}(g^*_\delta([\Phi(x_\delta, s)]_n) - g^*_\delta([\Phi^*_\delta(x_\delta, s)]_n))ds. \end{aligned} \tag{74}$$

Let us assume without loss of generality that $\underline{t}_\delta > \underline{t}_\delta^*$. To estimate the second integral in (74) we divide it into three parts:

$$\begin{aligned}
&- \quad b_1 \int_0^{\underline{t}_\delta^*} e^{-a_1(T_\delta - s)}(g_\delta^*([\Phi(x_\delta, s)]_n) - g_\delta^*([\Phi^*(x_\delta, s)]_n))ds \\
&- \quad b_1 \int_{\underline{t}_\delta^*}^{\underline{t}_\delta} e^{-a_1(T_\delta - s)}(g_\delta^*([\Phi(x_\delta, s)]_n) - g_\delta^*([\Phi^*(x_\delta, s)]_n))ds \qquad (75)\\
&- \quad b_1 \int_{\underline{t}_\delta}^{T_\delta} e^{-a_1(T_\delta - s)}(g_\delta^*([\Phi(x_\delta, s)]_n) - g_\delta^*([\Phi^*(x_\delta, s)]_n))ds.
\end{aligned}$$

Since functions $g_\delta^*(x)$ are uniformly bounded in the neighborhood of M the first two integrals are $O(\underline{t}_\delta^*)$ and $O(\underline{t}_\delta - \underline{t}_\delta^*)$ respectively. Observe, that by (59), there is δ'' such that for all $\delta \leq \delta''$ the functions $[\Phi(x_\delta, t)]_n$ and $[\Phi^*(x_\delta, t)]_n$ are monotone functions of $t \in [0, T_\delta(x_\delta)]$ and $t \in [0, T_\delta^*(x_\delta)]$ respectively. We use this fact to estimate the third integral. For $\delta \leq \delta''$ and $t > \underline{t}_\delta^* > \underline{t}_\delta$ we have that

$$[\Phi(x_\delta, t)]_n > M_n + \delta + \eta \qquad \text{and} \qquad [\Phi^*(x_\delta, t)]_n > M_n + \delta + \eta. \qquad (76)$$

Note that for such values of $[\Phi(x_\delta, t)]_n$ and $[\Phi^*(x_\delta, t)]_n$ the derivative $\frac{d}{dx}(g_\delta^*(\cdot))$ is bounded uniformly in δ and so

$$g_\delta^*([\Phi(x_\delta, s)]_n) - g_\delta^*([\Phi^*(x_\delta, s)]_n) = C([\Phi(x_\delta, s)]_n) - [\Phi^*(x_\delta, s)]_n = C[Y_\delta(s)]_n.$$

It follows from (57) that the third integral is

$$O(T_\delta^n - \underline{t}_\delta^n) \text{ as } \delta \to 0. \qquad (77)$$

This finishes the estimates of the second integral in (74).

Now we estimate the first integral in (74). We divide the integral again into three parts

$$\begin{aligned}
&- \quad b_1 \int_0^{\underline{t}_\delta^*} e^{-a_1(T_\delta - s)}(f([\Phi(x_\delta, s)]_n) - g_\delta^*([\Phi(x_\delta, s)]_n))ds \qquad (78)\\
&- \quad b_1 \int_{\underline{t}_\delta^*}^{\underline{t}_\delta} e^{-a_1(T_\delta - s)}(f([\Phi(x_\delta, s)]_n) - g_\delta^*([\Phi(x_\delta, s)]_n))ds \\
&- \quad b_1 \int_{\underline{t}_\delta}^{T_\delta} e^{-a_1(t - s)}(f([\Phi(x_\delta, s)]_n) - g_\delta^*([\Phi(x_\delta, s)]_n))ds.
\end{aligned}$$

Since the function $f - g_\delta^*$ is uniformly bounded in the neighborhood of M, the first and second integrals are $O(\underline{t}_\delta^*)$ and $O(\underline{t}_\delta - \underline{t}_\delta^*)$ respectively.

To estimate the third integral we use the intermediate value theorem to get

$$\begin{aligned}
&- \quad b_1 \int_{\underline{t}_\delta}^{T_\delta} e^{-a_1(t - s)}(f([\Phi(x_\delta, s)]_n) - g_\delta^*([\Phi(x_\delta, s)]_n))ds \\
&= \quad -b_1(T_\delta(x_\delta) - \underline{t}_\delta)(f(\chi_\delta) - g_\delta^*(\chi_\delta))
\end{aligned}$$

where $\chi_\delta \geq M_n + \delta + \eta$ for $\delta \leq \delta''$ by (76). Therefore $f(\chi_\delta) - g^*(\chi_\delta) \geq L$ by the definition of functions $\hat{g}$ and $\check{g}$.

Now we gather all the estimates of (74) together and get

$$\begin{aligned}|Y_\delta(T_\delta)]_1| &= O(\underline{t}^*_\delta) + O(\underline{t}_\delta - \underline{t}^*_\delta) + O(T^n_\delta - \underline{t}^n_\delta) + O(T_\delta(x_\delta) - \underline{t}_\delta) \\ &= O(\frac{\delta^k}{h^{\frac{1}{2}}(\delta)}) + O(h^{\frac{1}{2}}(\delta) - \frac{\delta^k}{h^{\frac{1}{2}}(\delta)})\end{aligned} \quad (79)$$

since $O(T_\delta(x_\delta)) = O(h(\delta)^{\frac{1}{2}})$ and (71). We have also ommited terms which are clearly of higher order. By assumption on k we get that

$$\limsup_{\delta \to 0} \frac{\frac{\delta^k}{h^{\frac{1}{2}}(\delta)}}{h^{\frac{1}{2}}(\delta)} = 0$$

and so from (79) it follows that

$$|c^*(x_\delta)| := |[Y_\delta(T_\delta(x_\delta))]_1| = O(T_\delta(x_\delta)) = O(h(\delta)^{\frac{1}{2}}) \text{ as } \delta \to 0.$$

The first statement of the Lemma follows now from (72), (73) and (57).

Observe, that from (74) and (79) it follows that there is δ''' such that for all $\delta \leq \delta'''$ we have

$$[Y_\delta(T_\delta(x_\delta))]_1 > 0 \text{ if } f(x) - g^*_\delta(x) < 0$$
$$[Y_\delta(T_\delta(x_\delta))]_1 < 0 \text{ if } f(x) - g^*_\delta(x) > 0.$$

The sign of the expression on the right depends on whether $g^*_\delta(x) = \check{g}_\delta(x)$ or $g^*_\delta(x) = \hat{g}_\delta(x)$. From the construction of $\check{g}_\delta(x)$ and $\hat{g}_\delta(x)$ we have that $-(f(x) - \check{g}_\delta(x)) = f(x) - \hat{g}_\delta(x) > 0$. Hence the second statement follows for $\delta \leq min\{\delta', \delta'', \delta''', \delta_4\}$.

Now we prove statement (b). Let $q_\delta \in U^x_\delta$ be a family with $d(\varphi_\delta(q_\delta)) =: l(\delta)$. By (30) for $\delta \leq \delta_4$ we have

$$l(\delta) \leq h(\delta). \quad (80)$$

Now we estimate $||\varphi^*_\delta(q_\delta) - \varphi_\delta(q_\delta)||$. From (72) and (73) we have

$$\varphi^*_\delta(q_\delta) - \varphi_\delta(q_\delta) = Y_\delta(T_\delta(q_\delta)) + O(l^{\frac{1}{2}}(\delta))o(l^{\frac{1}{2}}(\delta)).$$

By (57) the first component $[Y_\delta(T_\delta(q_\delta))]_1$ dominates the vector $Y_\delta(T_\delta(q_\delta))$. Note that from (74)

$$[Y_\delta(T_\delta(q_\delta))]_1 \leq O(T_\delta(q_\delta)) = O(l^{\frac{1}{2}}(\delta)) \text{ as } \delta \to 0$$

since both $g^*_\delta(x)$ and $f - g^*_\delta(x)$ are uniformly bounded in the neighborhood of M. Therefore, using (57) and (80) we can conclude

$$\begin{aligned}||\varphi^*_\delta(q_\delta) - \varphi_\delta(q_\delta)|| &\leq ||Y_\delta(T_\delta(q_\delta))|| + O(l^{\frac{1}{2}}(\delta))o(l^{\frac{1}{2}}(\delta)) \\ &\leq O(l^{\frac{1}{2}}(\delta)) + O(l^{\frac{1}{2}}(\delta))o(l^{\frac{1}{2}}(\delta)) \\ &\leq O(h^{\frac{1}{2}}(\delta)) + O(h^{\frac{1}{2}}(\delta))o(h^{\frac{1}{2}}(\delta)) \\ &= ||\varphi^*_\delta(x_\delta) - \varphi_\delta(x_\delta)|| + O(h(\delta))\end{aligned}$$

where in the last step we have used the statement 1(a) of this Lemma. This finishes the proof of the first part of the Lemma.

The proof of the second part is analogous to the proof of the first part. We let $Z_\delta(t) = \Phi(y_\delta, t) - \Phi^*_\delta(y_\delta, t)$ for a family $y_\delta \in H$ and, as in (74) and (57) we get

$$\begin{aligned}[Z_\delta(t)]_1 &= -b_1 \int_0^t e^{-a_1(t-s)}(f([\Phi(y_\delta,s)]_n) - g^*_\delta([\Phi(y_\delta,s)]_n))ds \\ &- b_1 \int_0^t e^{-a_1(t-s)}(g^*([\Phi(y_\delta,s)]_n) - g^*_\delta([\Phi^*(y_\delta,s)]_n))ds\end{aligned}$$

$$[Z_\delta(t)]_i = O(t^i) \text{ as } \delta \to 0.$$

We observe that $\psi_\delta(y_\delta) - \psi^*_\delta(y_\delta) = Z_\delta(\tau_\delta) + \Phi^*(y_\delta, \tau_\delta(y_\delta)) - \Phi^*(y_\delta, \tau^*_\delta(y_\delta))$. The analogous estimate as in the first part leads to

$$||\Phi^*(y_\delta, \tau_\delta(y_\delta)) - \Phi^*(y_\delta, \tau^*_\delta(y_\delta))|| = O(\iota^{\frac{1}{2}}(\delta))o(\iota^{\frac{1}{2}}(\delta)) \text{ as } \delta \to 0.$$

To estimate $[Z_\delta(\tau_\delta)]_1$ we decompose the corresponding integrals into three parts using the times m_δ, defined by $[\Phi_\delta(y_\delta, \tau_\delta - m_\delta)]_n = M_n + \delta + \eta$ and m^*_δ, defined by $[\Phi^*_\delta(y_\delta, \tau^*_\delta - m^*_\delta)]_n = M_n + \delta + \eta$ instead of the times $\underline{t}_\delta$ and $\underline{t}^*_\delta$. The rest of the proof is the same as the proof of the first part. □

4.3 Proof of Lemma 3.15

The main technical obstacle in the whole proof comes from the fact that the family u^*_δ does not have to be continuous and, although $u^*_\delta \to M$ as $\delta \to 0$, the limit $\lim_{\delta \to 0} \frac{d(u^*_\delta)}{\delta^k}$ does not have to exist.

We start with the following Claim.

Claim 1

$$\liminf_{\delta \to 0} \frac{||\varphi^*_\delta(z_\delta) - \varphi_\delta(z_\delta)||}{\delta} > 0.$$

Proof. We assume by contradiction that $\liminf_{\delta \to 0} \frac{||\varphi^*_\delta(z_\delta) - \varphi_\delta(z_\delta)||}{\delta} = 0$. We will show that this assumption implies

$$\liminf_{\delta \to 0} \frac{\alpha \cdot (\varphi^*_\delta(z_\delta) - M)}{\delta} > 0.$$

To that end we write

$$\alpha \cdot (\varphi^*_\delta(z_\delta) - M) = \alpha \cdot (\varphi^*_\delta(z_\delta) - \varphi_\delta(z_\delta)) + \alpha \cdot (\varphi_\delta(z_\delta) - M) \tag{81}$$

and estimate the right hand side. Using the cosine rule we get that $\alpha \cdot (\varphi^*_\delta(z_\delta) - \varphi_\delta(z_\delta)) = ||\alpha|| ||\varphi^*_\delta(z_\delta) - \varphi_\delta(z_\delta)|| \cos \beta'(\delta)$, where $\beta'(\delta)$ is the angle between α and $\varphi^*_\delta(z_\delta) - \varphi_\delta(z_\delta)$. Thus by the assumption we have

$$\liminf_{\delta \to 0} \frac{\alpha \cdot (\varphi^*_\delta(z_\delta) - \varphi_\delta(z_\delta))}{\delta} = 0.$$

Now $\alpha \cdot (\varphi_\delta(z_\delta) - M) = ||\varphi_\delta(z_\delta) - M|| \cos\beta(\delta)$ where $\beta(\delta)$ is the angle between α and $\varphi_\delta(z_\delta) - M$. Since $\cos\beta(\delta) = \alpha \cdot m + O(\delta^2)$ we have

$$\alpha \cdot (\varphi_\delta(z_\delta) - M) = O(||\varphi_\delta(z_\delta) - M||) \text{ as } \delta \to 0. \tag{82}$$

Since $TW^u_\Gamma(0) \cap H$ is tangent to $W^u \cap H$ it follows from (36) that for sufficiently small δ, we have $||\varphi_\delta(z_\delta) - M|| = O(\delta)$ as $\delta \to 0$. Hence from (81) and (82) we get that

$$\begin{aligned}\liminf_{\delta \to 0} \frac{\alpha \cdot (\varphi^*_\delta(z_\delta) - M)}{\delta} &\geq \\ \liminf_{\delta \to 0} \frac{\alpha \cdot (\varphi^*_\delta(z_\delta) - \varphi_\delta(z_\delta))}{\delta} &+ \liminf_{\delta \to 0} \frac{\alpha \cdot (\varphi_\delta(z_\delta) - M)}{\delta} > 0.\end{aligned}$$

On the other hand since $\varphi^*_\delta(z_\delta) \in W^s \cap H$ and $W^s \cap H$ is tangent to $TW^s_\Gamma(0) \cap H$ we have that

$$\alpha \cdot (\varphi^*_\delta(z_\delta) - M) = O(||\varphi^*_\delta(z_\delta) - M||^2).$$

By (46) $||\varphi^*_\delta(z_\delta) - M|| \leq O(\delta^{\frac{3}{4}})$ and since $\delta^{\frac{3}{2}} < \delta$ for sufficiently small δ, we get that

$$\limsup_{\delta \to 0} \frac{\alpha \cdot (\varphi^*_\delta(z_\delta) - M)}{\delta} = 0.$$

This is a contradiction and the Claim is proved. □

Now we use Lemma 3.12. Assume that z_δ is such that $d(\varphi^*_\delta(z_\delta)) = d(u^*_\delta) = l(\delta)$ with $\limsup_{\delta \to 0} \frac{\delta^k}{l(\delta)} = 0$ for some k. By Lemma 3.12 we have

$$||\varphi_\delta(z_\delta) - \varphi^*_\delta(z_\delta)|| = |c^*(z_\delta)| = O(l(\delta)^{\frac{1}{2}}). \tag{83}$$

Comparing to the Claim we get $\liminf_{\delta \to 0} \frac{l(\delta)^{\frac{1}{2}}}{\delta} > 0$ which implies

$$\limsup_{\delta \to 0} \frac{\delta}{l(\delta)^{\frac{1}{2}}} < \infty \quad \text{and} \quad \limsup_{\delta \to 0} \frac{\delta^2}{l(\delta)} < \infty.$$

Consequently,

$$\limsup_{\delta \to 0} \frac{\delta^k}{l(\delta)} = 0$$

for all $k > 2$ and Lemma is proved. □

4.4 Proof of Lemma 3.16

Recall that by assumption $z_\delta \in U^X_\delta$ is a family with $\varphi^*_\delta(z_\delta) = u^*_\delta \in W^s \cap H \cap \varphi^*_\delta(U^X_\delta)$ and $d(u^*_\delta) = l(\delta)$. The family X is a $(\frac{3}{2}, +)$-family and $\eta(\delta) = \delta^k$ with $k > 2$. We have shown in Lemma 3.15 that

$$\limsup_{\delta \to 0} \frac{\delta^k}{l(\delta)} = 0 \text{ for all } k > 2.$$

Therefore Lemma 3.12 applies and

$$u_\delta^* - M = \varphi_\delta(z_\delta) - M - c^*(z_\delta)e_1 + O(l(\delta)).$$

We approximate $\varphi_\delta(z_\delta)$ by the map defined by the linear flow $\bar{\varphi}(z_\delta)$ and get

$$u_\delta^* - M = \bar{\varphi}_\delta(z_\delta) - M - c^*(z_\delta)e_1 + O(l(\delta)) + O(||\bar{\varphi}_\delta(z_\delta) - M||^2) \tag{84}$$

where we have used the fact that $||\varphi_\delta(z_\delta) - M|| = O(||\bar{\varphi}_\delta(z_\delta) - M||^2)$ for sufficiently small δ. We denote $P_\delta := \bar{\varphi}_\delta(z_\delta)$. Since $u_\delta^* \in W^s \cap H$ there exists a point $R_\delta \in TW_\Gamma^s(0) \cap H$ such that $||u_\delta^* - M|| = ||R_\delta - M|| + O(||R_\delta - M||^2)$. Therefore

$$\alpha \cdot (u_\delta^* - M) = O(||u_\delta^* - M||^2)$$

as $\delta \to 0$ from which we obtain

$$[u_\delta^* - M]_1 = \frac{1}{[\alpha]_1}(-\sum_{i=2}^{n}[\alpha]_i[u_\delta^* - M]_i) + O(||u_\delta^* - M||^2). \tag{85}$$

Since $P_\delta \in TW_\Gamma^u(0) \cap H$ by the cosine law $\alpha \cdot (P_\delta - M) = K||P_\delta - M||$ with $K = m \cdot \alpha$. The vector m is the unit vector in $TW_\Gamma^u(0) \cap H$ (see Lemma 3.4). Therefore

$$[P_\delta - M]_1 = \frac{1}{[\alpha]_1}(K||P_\delta - M|| - \sum_{i=2}^{n}[\alpha]_i[P_\delta - M]_i). \tag{86}$$

Now we compare the components $2, \ldots n$ in (84) and get

$$-\sum_{i=2}^{n}[\alpha]_i[P_\delta - M]_i) = -\sum_{i=2}^{n}[\alpha]_i[u_\delta^* - M]_i) + O(l(\delta)) + O(||P_\delta - M||^2).$$

Using this result in (85) and then (86) we obtain

$$\begin{aligned}
[u_\delta^* - M]_1 &= \frac{1}{[\alpha]_1}(-\sum_{i=2}^{n}[\alpha]_i[P_\delta - M]_i) + O(l(\delta)) + O(||u_\delta^* - M||^2) + O(||P_\delta - M||^2) \\
&= [P_\delta - M]_1 - \frac{1}{[\alpha]_1}(K||P_\delta - M||) + O(l(\delta)) + O(||u_\delta^* - M||^2) + O(||P_\delta - M||^2) \\
&\leq ||P_\delta - M||(1 - \frac{K}{[\alpha]_1}) + O(l(\delta)) + O(||u_\delta^* - M||^2) + O(||P_\delta - M||^2) \qquad (87)
\end{aligned}$$

since $[P_\delta - M]_1 \leq ||P_\delta - M||$. Note that

$$||P_\delta - M|| = ||\Pi(w(s)) - M|| = \rho^u||w(s) - M|| \leq \rho^u\delta \tag{88}$$

and thus $[P_\delta - M]_i \leq m_i\rho^u\delta$ for $i = 1, \ldots, n$ where m_i is the i-th component of the unit vector m in $TW_\Gamma^u(0) \cap H$. It follows from (84) that

$$[u_\delta^* - M]_i \leq m_i\rho^u\delta + O(l(\delta)) + O((\rho^u\delta)^2) \tag{89}$$

for $i = 2, \ldots, n$.

Finally we compute $||u_\delta^* - M||$ using (87), (88) and (89):

$$\begin{aligned}
||u_\delta^* - M|| &\leq \sqrt{(\rho^u\delta(1 - \frac{K}{[\alpha]_1}))^2 + \sum_{i=2}^{n}(m_i\rho^u\delta)^2 + O(l(\delta)) + O(||u_\delta^* - M||^2) + O(\delta^2)} \\
&= \rho^u\delta\sqrt{(1 - \frac{K}{[\alpha]_1})^2 + \sum_{i=2}^{n} m_i^2 + O(l(\delta)) + O(||u_\delta^* - M||^2) + O(\delta^2)} \\
&= C\rho^u\delta + O(l(\delta)) + O(\delta^2) + O(\delta^2) \\
&\leq C\rho^u\delta + O(\delta^{\frac{3}{2}}) \qquad (90)
\end{aligned}$$

where in the last step we have used (47) to get $l(\delta) \leq O(\delta^{\frac{3}{2}})$ as $\delta \to 0$. This finishes first part of the proof.

Now we consider arbitrary family $x_\delta \in U_\delta^z$. We will show that $||\varphi_\delta^*(x_\delta) - M|| \leq (C + 2)\rho^u\delta + O(\delta^{\frac{3}{2}})$ as $\delta \to 0$.

Claim 2

$$||\varphi_\delta(x_\delta) - M|| \leq ||\varphi_\delta(z_\delta) - M|| + O(\delta^2).$$

Proof. We assume without loss that δ is sufficiently small so that $||\varphi_\delta(x_\delta) - M|| \leq ||\bar{\varphi}_\delta(x_\delta) - M|| + O(||\bar{\varphi}_\delta(x_\delta) - M||^2)$. Since $\bar{\varphi}_\delta$ is defined by the linearized flow and both x_δ and z_δ belong to the unstable manifold of $\gamma(t)$, for δ sufficiently small we have $||\bar{\varphi}_\delta(x_\delta) - M|| \leq ||\bar{\varphi}_\delta(z_\delta) - M||$. Therefore

$$\begin{aligned}
||\varphi_\delta(x_\delta) - M|| &\leq ||\bar{\varphi}_\delta(x_\delta) - M|| + O(||\bar{\varphi}_\delta(x_\delta) - M||^2) \\
&\leq ||\bar{\varphi}_\delta(z_\delta) - M|| + O(||\bar{\varphi}_\delta(x_\delta) - M||^2) \\
&= ||\varphi_\delta(z_\delta) - M|| + O(||\bar{\varphi}_\delta(x_\delta) - M||^2) + O(||\bar{\varphi}_\delta(z_\delta) - M||^2) \\
&\leq ||\varphi_\delta(z_\delta) - M|| + O(\delta^2)
\end{aligned}$$

where in the last step we used (36) to obtain $||\bar{\varphi}_\delta(x_\delta) - M|| = O(\delta)$ as $\delta \to 0$. □

Now we finish the proof of the Lemma. Using Claim 2, Lemma 3.12.1(b) and repeatedly the triangular inequality we get

$$\begin{aligned}
||\varphi_\delta^*(x_\delta) - M|| &\leq ||\varphi_\delta^*(x_\delta) - \varphi_\delta(x_\delta)|| + ||\varphi_\delta(x_\delta) - M|| \\
&\leq ||\varphi_\delta^*(z_\delta) - \varphi_\delta(z_\delta)|| + O(h(\delta)) + ||\varphi_\delta(z_\delta) - M|| + O(\delta^2) \\
&\leq ||\varphi_\delta^*(z_\delta) - M|| + ||\varphi_\delta(z_\delta) - M|| + ||\varphi_\delta(z_\delta) - M|| + O(\delta^2) + O(h(\delta)) \\
&\leq (C + 2)\rho^u\delta + O(\delta^2) + O(\delta^{\frac{3}{2}})
\end{aligned}$$

where in the last step we have used (47), (90) and the fact (see (88)) that

$$||\varphi_\delta(z_\delta) - M|| = ||P_\delta - M|| + O(||P_\delta - M||^2) \leq \rho^u\delta + O(\delta^2).$$

□

5 Proof of Theorem 1.13

In order to prove the Theorem 1.13 we will compute a value b_1^* for which the linearization of (8) at the origin has a pair of purely imaginary eigenvalues and then we will average the nonlinear terms in (8) to determine if the associated Hopf bifurcation is subcritical or supercritical.

Our discussion of the averaging will follow the paper by Chow and Mallet-Paret [3].

5.1 Averaging.

Let us fix $a_i > 0,\ i = 1, \dots, n,\ b_i > 0,\ i = 2, \dots, n$ and n. Suppose that there exists a $b_1^* = b_1(a_i, b_i, n)$ such that if we denote by

$$\dot{x} = Ax$$

the linearization of (8) at the origin with $b_1 = b_1^*$, then

1. A has a pair of purely complex eigenvalues $\pm iu$
2. other eigenvalues have nonzero real parts
3. the complex pair crosses the imaginary axis with nonzero speed.

These are the assumptions for a generic Hopf bifurcation at b_1^*.

We will review first the averaging method and later we will prove that such a b_1^* exists. Consider new coordinates $z = (w_1, w_2, y_1, \dots, y_{n-2}) \in R^n$, where w_1, w_2 are eigenvectors corresponding to $\pm iu$, $y_1, \dots, y_{n-2}$ are perpendicular to w_1 and w_2 and $span\{y_1, \dots, y_{n-2}\} \oplus span\{w_1, w_2\} = R^n$. We will denote $w = (w_1, w_2)$ and $y = (y_1, \dots, y_{n-2})$. Following [3] in this coordinates we can write our system as follows:

$$\begin{aligned}
\dot{w} &= \alpha E(\alpha)y + F(y,\alpha)y^2 + A_P(\alpha)w + G(y,\alpha)wy + \\
&\quad + B_2(y,\alpha)w^2 + B_3(y,\alpha)w^3 + \dots \\
\dot{y} &= \alpha H(\alpha)w + J(w,\alpha)w^2 + A_Q y + \alpha M(\alpha)y + N(w,\alpha)wy + \\
&\quad + \Gamma_2(w,\alpha)y^2 + \Gamma_3(w,\alpha)y^3 + \dots
\end{aligned}$$

Changing to the polar coordinates $w = (r\cos\theta, r\sin\theta)$ we get:

$$\begin{aligned}
\dot{r} &= [\alpha E_1(\theta\alpha)y + F_1(\theta,y,\alpha)y^2] + r[\alpha + G_2(\theta,y,\alpha)y] + \\
&\quad + r^2 C_3(\theta,y,\alpha) + r^3 C_4(\theta,y,\alpha) + \dots \qquad (91) \\
\dot{\theta} &= \frac{1}{r}[\alpha E_1^*(\theta\alpha)y + F_1^*(\theta,y,\alpha)y^2] + [\omega(\alpha) + G_2^*(\theta,y,\alpha)y] + \\
&\quad + rD_3(\theta,y,\alpha) + r^2 D_4(\theta,y,\alpha) + \dots \\
\dot{y} &= \text{as above but with } w = (r\cos\theta, r\sin\theta)
\end{aligned}$$

Here $\alpha \in (b_1^* - \delta, b_1^* + \delta)$ is the bifurcation parameter close to the bifurcation value and subscripts in $C_2, C_3, \dots$ indicate that the given function f_j is homogeneous of degree j in $(\cos\theta, \sin\theta)$.

Also observe that

$$C_j(\theta,y,\alpha)=\cos\theta B^1_{j-1}(\cos\theta,\sin\theta,y,\alpha)+\sin\theta B^2_{j-1}(\cos\theta,\sin\theta,y,\alpha)$$

$$D_j(\theta,y,\alpha)=\cos\theta B^2_{j-1}(\cos\theta,\sin\theta,y,\alpha)-\sin\theta B^1_{j-1}(\cos\theta,\sin\theta,y,\alpha)$$

Scaling $r\to\epsilon r,\ y\to\epsilon y,\alpha\to\epsilon\alpha$ we get

$$\begin{aligned}
\dot r &= \epsilon[\alpha r+r^2C_3(\theta,\epsilon y,\epsilon\alpha)+F_1(\theta,\epsilon y,\epsilon\alpha)y^2+rG_2(\theta,\epsilon y,\epsilon\alpha)y]\\
&\quad +\epsilon^2r^3C_4(\theta,\epsilon y,\epsilon\alpha)+O(\epsilon^3)\\
\dot\theta &= \omega_0+\epsilon[\alpha\omega'(0)+rD_3(\theta,\epsilon y,\epsilon\alpha)+\frac{\alpha}{r}E_1^*(\theta,\epsilon\alpha)y+\\
&\quad +\frac{1}{r}F_1^*(\theta,\epsilon y,\epsilon\alpha)y^2+G_2^*(\theta,\epsilon y,\epsilon\alpha)y]+O(\epsilon^2)\\
\dot y &= A_Qy+\epsilon[\alpha H(\alpha)w+J(\epsilon w,\epsilon\alpha)w^2+\alpha M(\epsilon\alpha)y+\\
&\quad +N(\epsilon w,\epsilon\alpha)+\Gamma_2(\epsilon w,\epsilon\alpha)y^2]+O(\epsilon^2)
\end{aligned}\tag{92}$$

Theorem 5.1 (Theorem 5.1,[3]) *Given system (92) there exists a coordinate change*

$$\bar r=r+\epsilon u_1(r,\theta,\alpha,\epsilon)+\epsilon z(r,\theta,\alpha,\epsilon)y+\epsilon^2u_2(r,\theta,\alpha,\epsilon)$$

transforming (92) into the averaged system of the form

$$\begin{aligned}
\dot{\bar r} &= \epsilon\alpha\bar r+\epsilon^2\bar r^3K+O(\epsilon|y|^2)+O(\epsilon^2|y|)+O(\epsilon^3)\\
\dot\theta &= \omega_0+O(\epsilon)\\
\dot y &= A_Qy+O(\epsilon)
\end{aligned}\tag{93}$$

where $K=K^*+K^{**}$ is a constant with

$$\begin{aligned}
K^* &= \frac{1}{2\pi}\int_0^{2\pi}C_4(\theta,0,0)-\frac{1}{\omega_0}C_3(\theta,0,0)D_3(\theta,0,0)d\theta\\
K^{**} &= \frac{1}{2\pi}\int_0^{2\pi}w^*(\theta)J(0,0)(cos\theta,sin\theta)^2d\theta,
\end{aligned}\tag{94}$$

where w^* is the unique 2π-periodic solution of

$$G_2(\theta,0,0)+w^*(\theta)\omega_0+w^*(\theta)A_Q=0.$$

Recall that for each (w,α), $J(w,\alpha)$ is a bilinear form in the w-space R^2 taking values in the y-space; in the theorem $J(0,0)$ acts on the point $(\cos\theta,\sin\theta)\in R^2$.

Theorem 5.2 (Theorem 6.2,[3]) *Given (93), then in the original unaveraged, unscaled system there is a unique periodic solution bifurcating from the origin either for*

$$\alpha>0\ \ if\ \ K<0$$

or

$$\alpha<0\ \ if\ K>0.$$

In order to prove Theorem 1.13 we need to show that $K > 0$. To do so we first need to identify functions $C_3, C_4, D_3, J, G_2, ..$ in our system and then compute K.

Let us rewrite (8) as

$$\dot{x} = Ax + f(x) + g(x)$$

where $g(x) = o(x^3)$, $f(x) = (-dx_n^3, 0, \ldots, 0)^T$.

Let U be a matrix with $w_1, w_2, y_1, \ldots, y_{n-2}$ as columns. Then

$$\dot{z} = U^{-1}AUz + U^{-1}f(Uz) \tag{95}$$

where $z = (w_1, w_2, y_1, \ldots, y_{n-2})^T$. Here we suppressed $g(x)$, because the averaging depends only on the terms up to the order 3. Now

$$f(Uz) = (-d(u_1w_1 + u_2w_2 + u \cdot y)^3, 0, 0, \ldots, 0)^T$$

where u_1, u_2 are constants, u is n-2-vector of constants and (u_1, u_2, u) is the n-th row vector of U.

Then

$$U^{-1}f(Uz) = \begin{pmatrix} -dp_1(u_1w_1 + u_2w_2 + u \cdot y)^3 \\ -dp_2(u_1w_1 + u_2w_2 + u \cdot y)^3 \\ -dp(u_1w_1 + u_2w_2 + u \cdot y)^3 \end{pmatrix}$$

where again p_1, p_2 are constants, p is n-2 vector of constants and (p_1, p_2, p) is the first column vector of U^{-1}.

Changing to polar coordinates $w_1 = r\cos\theta,\ w_2 = r\sin\theta$ and comparing with (92) we get

$$\begin{aligned} C_4 &= (\theta, y, 0) = -d(p_1\cos\theta + p_2\sin\theta)(u_1\cos\theta + u_2\sin\theta)^3 \\ C_3 &= (\theta, y, 0) = -u \cdot yd(3p_1\cos\theta + 3p_2\sin\theta)(u_1\cos\theta + u_2\sin\theta)^2 \\ D_3(\theta, y, 0) &= -d(p_2\cos\theta - p_1\sin\theta)(u_1\cos\theta + u_2\sin\theta)^3 \\ J(w, 0) &= J(r, \theta, 0) = -pdr(u_1\cos\theta + u_2\sin\theta) \end{aligned}$$

Since $J(0,0) = 0$ and $C_3(\theta, 0, 0) = 0$ we have from (94)

$$\begin{aligned} K &= K_* = \frac{1}{2\pi}\int_0^{2\pi} C_4(\theta, 0, 0)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} -d(p_1\cos\theta + p_2\sin\theta)(u_1\cos\theta + u_2\sin\theta)^3 d\theta \\ &= -\frac{3\pi}{4}d(p_1u_1 + p_2u_2)(u_1^2 + u_2^2). \end{aligned}$$

In order to prove Theorem 1.13 we need to compute constants p_1, p_2, u_1, u_2 and so we need to look for the eigenvectors corresponding to $\pm iu$. Indeed, if w_1, w_2 are eigenvectors corresponding to $\pm iu$ at $\alpha = 0$, then u_1 is the n-th entry in w_1 and u_2 is the $n-$th entry in w_2.

Also note that $p_1 = U^{-1}[1,1],\ \ p_2 = U^{-1}[2,1]$ where $A[i,j]$ is the element in $i-$th row and $j-$th column of the matrix A.

5.2 Eigenvalues and eigenvectors.

In this section we prove that there is $b_1^* = b_1(a_i, b_i, n)$ such that at this value a generic Hopf bifurcation from the origin takes place. We show that all eigenvalues except the bifurcating pair have negative real parts at the moment of the bifurcation. Besides verifying one of the assumptions of the generic Hopf bifurcation, this will also show that in subcritical case the bifurcating periodic orbit will have a one dimensional unstable manifold.

Finally, we compute constant the K and show that $K > 0$ which will show that the Hopf bifurcation is subcritical.

Lemma 5.3 *There is a value $b_1^* = b_1(a_i, b_i, n)$ such that a generic Hopf bifurcation from the origin takes place. Furthermore, all eigenvalues except the bifurcating pair have a negative real part at the moment of the bifurcation.*

Proof. Let us start with the computation of the eigenvalues. Linearization of the system (8) at the origin is

$$A = \begin{bmatrix} -a_1 & 0 & \dots & 0 & -b_1 \\ b_2 & -a_2 & \dots & 0 & 0 \\ & & \vdots & & \\ 0 & & \dots & b_n & -a_n \end{bmatrix}$$

and its characteristic polynomial is

$$\prod_{j=1}^{n}(-a_j - \lambda) + (-1)^n \prod_{j=1}^{n} b_j = 0$$

or

$$\prod_{j=1}^{n}(a_j + \lambda) + \prod_{j=1}^{n} b_j = 0 \tag{96}$$

Let us put $a_j + \lambda = m_j e^{i\alpha_j}$ where $m_j = |a_j + \lambda|$. Then (96) becomes

$$(\prod_{j}^{n} m_j) e^{i\sum \alpha_j} = -\prod_{j=1}^{n} b_j$$

and so we have

$$\begin{aligned} \sum_j \alpha_j &= s\pi \quad s = \pm 1, \pm 3, \dots, l \\ \prod_j m_j &= \prod_j b_j \end{aligned} \tag{97}$$

where $l = n$ if n is odd and $l = \pm(n-1)$ if n is even.

Denote $T := \prod_j b_j$. Since $b_2, \dots, b_n$ are fixed and b_1 is a bifurcation parameter we can treat T as a bifurcation parameter as well.

Since $m_j = |a_j + \lambda|$ then $\cos\alpha_j = \frac{Re\,(a_j+\lambda)}{m_j}$ and $\sin\alpha_j = \frac{Im\,(a_j+\lambda)}{m_j} = \frac{Im\,\lambda}{m_j}$ since a_j is real. Simple observation shows that $\alpha_j(\lambda)$ is the angle between the real axis and the half line starting at the point $(-a_j, 0)$ and going through the point λ in the complex plane.

Let $\{g_s(T)\}$, $T \in [0, \infty]$, $s = \pm 1, \pm 3, \ldots, l$, l defined as above, be a collection of curves in the complex plane such that if

$$y \in g_s(\cdot) \text{ then}$$

$$\sum_{j=1}^{n} \alpha_j(y) = s\pi. \tag{98}$$

By (97) all eigenvalues for all possible values of T lie on these curves (Figure 15). On the other hand for a given $\bar{T}$ all eigenvalues form a collection

$$\{g_s(\bar{T})\}, \quad s = \pm 1, \pm 3, \ldots, l.$$

It is easy to see that if $g_s(T)$ is not a real number then

$$g_{-s}(T) = \overline{g_s(T)} \tag{99}$$

where $\overline{g}$ is complex conjugate of g.

Asymptotes as $T \to \infty$. If $T \to \infty$ also $|\lambda| \to \infty$ and we have asymptotically

$$(a_1 + \lambda)(a_2 + \lambda) \ldots (a_n + \lambda) \approx (\lambda + v)^n.$$

To compute v we expand both sides

$$\lambda^n + \lambda^{n-1}(\sum_{j=1}^{n} a_j) + \ldots \approx \lambda^n + \lambda^{n-1}(nv) + \ldots$$

and so v is given by

$$v = \frac{1}{n} \sum_{j=1}^{n} a_j.$$

Let $v_s = \{z \in C \,|\, arg(z - v) = s\frac{\pi}{n},\ s = \pm 1, \pm 3, \ldots, l\}$ be a half-line in the complex plane. One can see that $g_s(T) \to v_s$ as $T \to \infty$ and if $n \geq 3$ then there is a parameter value T_1 such that $g_{\pm 1}(T_1) = \pm i q_1$ for some q_1. This correspond to a Hopf bifurcation for the parameter value T_1.

Claim 3 *Let $\tau = \{T \,|\, g_{\pm j}(T) \in$ imaginary axis for some $j\}$. Then $T_1 = \min_{T \in \tau} T$.*

Proof. Let us denote imaginary axis by Im. Let us take $ix, iy \in Im$ and because of symmetry (99) we will restrict our attention to $x, y > 0$.

If $x > y$ then $\sum_j \alpha_j(ix) > \sum_j \alpha_j(iy)$ where $\alpha_j(\lambda)$ was defined above. Observe, that also the opposite implication holds.

Let us denote $\pm i q_s$ the purely imaginary pair of complex numbers such that $\sum_j \alpha_j(i q_s) = s\pi$. Then by the preceding argument $q_j > q_k$ if $j > k$ and , in particular,

$$q_1 < q_k \text{ for all } k > 1. \tag{100}$$

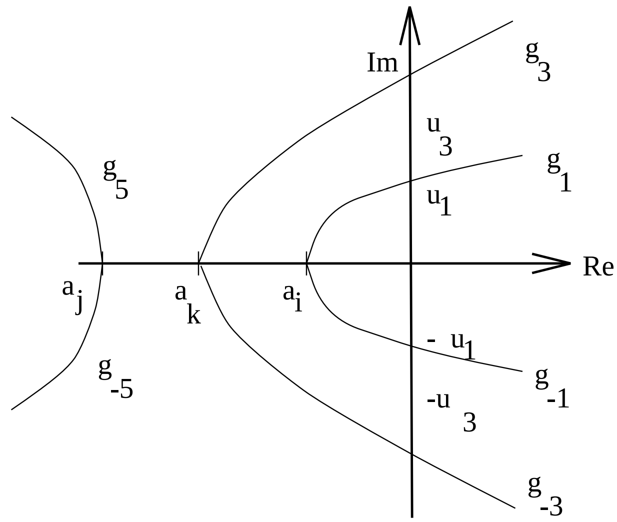


Figure 15: Branches of eigenvalues parameterized by T.

Now taking $\lambda = iq_s$ in (96) we let

$$T_s := \prod_{j=1}^{n} |a_j + iq_s| = \prod_{j=1}^{n} \sqrt{a_j^2 + q_s^2} \quad \text{for } s = \pm 1, \pm 3 \ldots, l.$$

Now (100) implies $T_1 < T_k$ for $k > 1$ as asserted. □

Note, that the pair $\pm iu := \pm iq_1$ are eigenvalues corresponding to the smallest $T = T_1$ and hence to the smallest bifurcation parameter b_1. Let us define b_1^* to be the value of b_1 such that $T_1 = b_1^* b_2 \ldots b_n$. Since $b_2, \ldots, b_n$ are fixed, we see that for all $b_1 < b_1^*$ the origin is stable and so at the bifurcation value all other eigenvalues have negative real part.

Now we show that the eigenvalues $\pm iu$ cross imaginary axis with nonzero speed. Let us parameterize the curve g_1 by $y \in \mathbf{R}$ i.e. $g_1 = g_1(y)$. By (98) we have

$$\frac{d}{dy} \sum_{j=1}^{n} \alpha_j(y) = 0. \tag{101}$$

We set $y := iu$. Observe that

$$\sum_{j=1}^{n} \alpha_j(iu) < \sum_{j=1}^{n} \alpha_j(i(u+\epsilon))$$

for ϵ small and so directional derivative of $\sum_{j=1}^{n} \alpha_j(y)$ at $y = iu$ in the positive direction of imaginary axis is positive. Similarly, one can easily check that the directional derivative of the sum in the positive direction of the real axis is negative. Therefore by (101) we see that

$$\frac{dg}{dy}|_{y=iu} \tag{102}$$

is a vector pointing into the first orthant (Figure 15).

Now we take $y = y(T)$ and examine $\prod_j m_j(y) = T$. We take a derivative and get

$$(\sum_{j=1}^{n} \frac{dm_j(y)}{dy} \prod_{i \neq j} m_i(y)) \frac{dy}{dT} = 1$$

from which

$$\frac{dy}{dT}|_{T=T_1} = \frac{1}{(\sum_{j=1}^{n} \frac{dm_j(y)}{dy} \prod_{i \neq j} m_i(y))|_{T_1}} \neq 0.$$

This together with (102) concludes the proof for $y = iu$. Same conclusion obviously holds for $y = -iu$. □

Now we compute the constants u_1, u_2, p_1, p_2 to determine the sign of K and thus the character of the Hopf bifurcation at $\pm iu$.

Lemma 5.4 *The eigenvectors $w_1 = (w_{11}, \ldots, w_{1n})$ and $w_2 = (w_{21}, \ldots, w_{2n})$ corresponding to the complex pair of eigenvalues $\pm iu$ can be chosen in such a way, that $u_1 = w_{1n} = 1$ and $u_2 = w_{2n} = 0$. Moreover $p_1 < 0$.*

We postpone the proof of the Lemma and prove Theorem 1.13.

Proof of Theorem 1.13. Recall that $K = -\frac{3\pi}{4} d(p_1 u_1 + p_2 u_2)(u_1^2 + u_2^2)$ and that by Theorem 5.2 we need to prove that $K > 0$.

Using Lemma 5.4 we have

$$K = -\frac{3\pi}{4} d p_1$$

and since $d > 0$ $sign\ (K) = -sign\ (p_1)$. By Lemma 5.4 $p_1 < 0$ and so $K > 0$. □

Proof of Lemma 5.4. Let us now turn to the computation of the (complex) eigenvector V corresponding to the pair of eigenvalues $\pm iu$.

Claim 4 $V = (k_2 e^{i(\alpha'_2 + \ldots + \alpha'_n)}, k_3 e^{i(\alpha'_3 + \ldots + \alpha'_n)}, \ldots, k_n e^{i\alpha'_n}, 1)$ *is an eigenvector for $\lambda = iu$, where*

$$k_i = \frac{m_i}{b_i} \frac{m_{i-1}}{b_{i-1}} \cdots \frac{m_n}{b_n}$$

and $\alpha'_j := \alpha_j(iu)$.

Proof. Let $B := A - (iu)I$ where I is the identity matrix. Since V is in the null space of B, V is perpendicular to all rows of matrix B. We set the n-th element of V to be 1 and then compute the $n-1$-th element from the requirement that n-th row of A is perpendicular to V. By induction we compute the $i-1$ element from the assumption that i-th row is perpendicular to V. The assertion follows by direct computation. □

Let us define

$$w_1 = Re\, V, \quad w_2 = Im\, V$$

to be the real and imaginary part of the vector V. Then $w_{1n} = 1$ and $w_{2n} = 0$ as claimed in the Lemma 5.4.

Now we proceed to prove that $p_1 < 0$. Recall that $p_1 = U^{-1}[1,1]$ and $p_2 = U^{-1}[2,1]$ and so

$$e_1 = p_1 w_1 + p_2 w_2 + py$$

where e_1 is the first standard unit vector.

Then

$$w_1 e_1 = p_1 w_1 w_1 + p_2 w_2 w_1$$

$$w_2 e_1 = p_1 w_1 w_2 + p_2 w_2 w_2$$

because y is perpendicular to span $\{w_1, w_2\}$.

Solving for p_2 from the second equation and plugging into the first one finds

$$p_1 = \frac{w_1 e_1 - w_2 e_1 \frac{w_1 w_2}{||w_2||^2}}{||w_1||^2 - \frac{(w_1 w_2)^2}{||w_2||^2}}.$$

Since w_1 and w_2 are linearly independent, by the Schwartz inequality

$$||w_1||^2 - \frac{(w_1 w_2)^2}{||w_2||^2} > 0.$$

If we show that the numerator

$$Q := w_1 e_1 - w_2 e_1 \frac{w_1 w_2}{||w_2||^2} < 0 \tag{103}$$

we will have $p_1 < 0$.

Let us prove (103). Our choice of w_1, w_2 implies $w_1 e_1 = k_2 \cos(\alpha'_2 + \ldots + \alpha'_n)$, $w_2 e_1 = k_2 \sin(\alpha'_2 + \ldots + \alpha'_n)$ and so

$$M := \frac{w_1 w_2}{|w_2|^2} = \frac{k_2^2 \cos(\alpha'_2 + \ldots + \alpha'_n)\sin(\alpha'_2 + \ldots + \alpha'_n) + \ldots + k_n^2 \cos\alpha'_n \sin\alpha'_n}{k_2^2 \sin^2(\alpha'_2 + \ldots + \alpha'_n) + \ldots + k_n^2 \sin^2\alpha'_n}.$$

To simplify notation let us denote $\beta_j = \alpha'_j + \ldots + \alpha'_n$ and $h_j = k_j^2$.

Lemma 5.5 *$M >$ cotg β_2*

Proof. Since $\alpha'_j = \alpha(iu)$ and $\sum_{j=1}^n \alpha'_j = \pi$ if $n \geq 3$ we have $\alpha'_j < \frac{\pi}{2}$ for all j. Therefore $\beta_2 = \pi - \alpha'_1 \in (\frac{\pi}{2}, \pi)$ and so

$$\cos\beta_2 < 0, \quad \sin\beta_2 > 0.$$

Let r be such an index that $\cos\beta_r < 0$ and $\cos\beta_{r+1} > 0$. We estimate

$$\begin{aligned} M &= \frac{\sum_{j=2}^n h_j \cos\beta_j \sin\beta_j}{\sum_{j=2}^n h_j \sin^2\beta_j} > \frac{\sum_{j=2}^r h_j \cos\beta_j \sin\beta_j}{\sum_{j=2}^n h_j \sin^2\beta_j} \\ &> \frac{\sum_{j=2}^r h_j \cos\beta_j \sin\beta_j}{\sum_{j=2}^r h_j \sin^2\beta_j} = \frac{\frac{1}{\sin\beta_2}}{\frac{1}{\sin\beta_2}} \frac{\sum_{j=2}^r h_j \cos\beta_j \sin\beta_j}{\sum_{j=2}^r h_j \sin^2\beta_j} =: L \end{aligned}$$

where the second inequality holds because the fraction is negative.

Since $\cos\beta_j > \cos\beta_2$ and $\frac{\sin\beta_j}{\sin\beta_2} > 1$ for $j = 3, \ldots, r$ we have

$$L \geq \frac{\sum_{j=2}^{r} h_j cotg\, \beta_2 \sin\beta_j}{\sum_{j=2}^{r} h_j \sin\beta_j} = cotg\beta_2$$

□

Now we can finish the proof of the Lemma 5.4 and thus finish the proof of the Theorem 1.13 by showing that $p_1 < 0$. Indeed, by the choice of w_1, w_2

$$Q = k_2 \cos\beta_2 - k_2 \sin\beta_2 M <$$

$$< k_2 \cos\beta_2 - k_2 \sin\beta_2 cotg\, \beta_2 = 0$$

by Lemma 5.5 and so $Q < 0$. By (103) $p_1 < 0$. □

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