First Test Solution, MATH 224, Fall 2006 (slightly modified)

1. (5 pts) Which of the points $(0, 1, 2)$, $(3, 4, 1)$, $(-2, 0, -3)$, $(1, -1, 2)$ is closest to the xy-plane? Which point is in the xz-plane?

$(3, 4, 1)$ is closest to the xy-plane, $(-2, 0, -3)$ is in the xz-plane.

2. (5 pts) Which of the following expressions are meaningful? Which are meaningless? For the meaningful ones, is the result a vector or a scalar?

- $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ - vector
- $(\mathbf{a} \cdot \mathbf{b}) + \mathbf{c}$ - meaningless
- $(|\mathbf{a}|\mathbf{b}) \cdot \mathbf{c}$ - scalar
- $|\mathbf{a} + \mathbf{b}| \times \mathbf{c}$ - meaningless
- $(\mathbf{a} - \mathbf{b}) \times ((\mathbf{a} \cdot \mathbf{b})\mathbf{b})$ - vector

3. (10 pts) Which of the following statements are true, which are false?

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ - true
- $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$ - false
- $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \mathbf{0}$ - true
- $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}'(t)$ - false
- $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{u}(t)) = \mathbf{0}$ - true

4. (10 pts) What is the dot product of two vectors with lengths 3 and 4, if the angle between them is $30°$? If we know in addition that both vectors lie in the xy-plane, what can be said about their cross product?

The dot product is $12 \cos 30° = 6\sqrt{3}$, the cross product has length $12 \sin 30° = 6$ and points in the direction of the positive or negative z-axis, so it is either $(0, 0, 6)$ or $(0, 0, -6)$.

5. (10 pts) Write down an equation for the plane which contains the points $(1, 2, 3)$, $(2, 3, 4)$, and $(3, 4, 6)$. Which of the points $(0, 1, 2)$ and $(0, 2, 1)$ lies in this plane?

Denoting the points by P, Q, and R, we get a normal vector by $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1, 1, 1 \rangle \times \langle 2, 2, 3 \rangle = \langle 1, -1, 0 \rangle$, so the equation for the plane is $(x - 1) - (y - 2) = 0$, or $x - y = -1$. The point $(0, 1, 2)$ is in the plane, $(0, 2, 1)$ is not.

6. (10 pts) What can you say about a planar curve if the curvature satisfies $\kappa = 0$ everywhere along the curve? What if $\kappa = 2$ instead?

If the curvature is 0, the curve lies on a straight line. If it is 2, it is on a circle of radius 1/2.

7. (20 pts) A river flowing east is 10m wide, and the water speed in the river is given by the function $f(x) = \frac{1}{5}x(10-x)$ (in m/s), where x is the distance from the north bank in meters. A boat proceeds with a constant speed of 2 m/s from a point A on the north bank, heading straight south. How far down the river will the boat arrive on the south bank?

After t seconds the boat is $x = 2t$ meters from the north bank. After 5 seconds the boat arrives at the other side. The water speed at time t is $f(x) = f(2t) = \frac{2t(10-2t)}{5}$, so the velocity vector is $\mathbf{v}(t) = \langle 2, \frac{2t(10-2t)}{5} \rangle = \langle 2, 4t - \frac{4}{5}t^2 \rangle$. The position vector at time $t = 5$ is $\mathbf{r}(5) = \int_0^5 \langle 2, 4t - \frac{4}{5}t^2 \rangle dt = \langle 2t, 2t^2 - \frac{4}{15}t^3 \rangle \Big|_0^5 = \langle 10, 50 - \frac{100}{3} \rangle = \langle 10, \frac{50}{3} \rangle$. So the answer is $50/3 = 16.67$ meters.

8. (15 pts) A wagon is pulled a distance of 50 m by a constant force of 20 N. The handle of the wagon is held at an angle of 45°. How much work is done?

$W = |\mathbf{F}||\mathbf{D}|\cos\alpha = 20 \cdot 50 \cos 45° = \frac{1000}{\sqrt{2}}$

9. (15 pts) Find the length of the curve $\mathbf{r}(t) = \left\langle 3t+1, 4t^{3/2} - 1, 3t^2 \right\rangle$, $0 \le t \le 1$.

$\mathbf{r}'(t) = \langle 3, 6t^{1/2}, 6t \rangle$, so $L = \int_0^1 |\mathbf{r}'(t)| dt = \int_0^1 \sqrt{9 + 36t + 36t^2}\, dt = 3\int_0^1 \sqrt{1 + 4t + 4t^2}\, dt = 3\int_0^1 (1+2t) dt = 3\left[t + t^2\right]_0^1 = 6.$