First Test Solution, MATH 224, Fall 2006 (slightly modified)

1. (5 pts) Which of the points \((0, 1, 2), (3, 4, 1), (-2, 0, -3), (1, -1, 2)\) is closest to the \(xy\)-plane? Which point is in the \(xz\)-plane?

(3, 4, 1) is closest to the \(xy\)-plane, \((-2, 0, -3)\) is in the \(xz\)-plane.

2. (5 pts) Which of the following expressions are meaningful? Which are meaningless? For the meaningful ones, is the result a vector or a scalar?

- \((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}\) - vector
- \((\mathbf{a} \cdot \mathbf{b}) + \mathbf{c}\) - meaningless
- \((|\mathbf{a}| \mathbf{b}) \cdot \mathbf{c}\) - scalar
- \(|\mathbf{a} + \mathbf{b}| \times \mathbf{c}\) - meaningless
- \((\mathbf{a} - \mathbf{b}) \times ((\mathbf{a} \cdot \mathbf{b}) \mathbf{b})\) - vector

3. (10 pts) Which of the following statements are true, which are false?

- \(\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}\) - true
- \(\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}\) - false
- \((\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0\) - true
- \(\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}'(t)\) - false
- \(\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{u}(t)) = 0\) - true

4. (10 pts) What is the dot product of two vectors with lengths 3 and 4, if the angle between them is \(30^\circ\)? If we know in addition that both vectors lie in the \(xy\)-plane, what can be said about their cross product?

The dot product is \(12 \cos 30^\circ = 6\sqrt{3}\), the cross product has length \(12 \sin 30^\circ = 6\) and points in the direction of the positive or negative \(z\)-axis, so it is either \((0, 0, 6)\) or \((0, 0, -6)\).

5. (10 pts) Write down an equation for the plane which contains the points \((1, 2, 3), (2, 3, 4),\) and \((3, 4, 6)\). Which of the points \((0, 1, 2)\) and \((0, 2, 1)\) lies in this plane?

Denoting the points by \(P, Q,\) and \(R\), we get a normal vector by \(\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = (1, 1, 1) \times (2, 2, 3) = (1, -1, 0)\), so the equation for the plane is \((x - 1) - (y - 2) = 0\), or \(x - y = 1\). The point \((0, 1, 2)\) is in the plane, \((0, 2, 1)\) is not.
6. (10 pts) What can you say about a planar curve if the curvature satisfies \( \kappa = 0 \) everywhere along the curve? What if \( \kappa = 2 \) instead?

If the curvature is 0, the curve lies on a straight line. If it is 2, it is on a circle of radius \( \frac{1}{2} \).

7. (20 pts) A river flowing east is 10m wide, and the water speed in the river is given by the function \( f(x) = \frac{1}{5}x(10 - x) \) (in m/s), where \( x \) is the distance from the north bank in meters. A boat proceeds with a constant speed of 2 m/s from a point \( A \) on the north bank, heading straight south. How far down the river will the boat arrive on the south bank?

After \( t \) seconds the boat is \( x = 2t \) meters from the north bank. After 5 seconds the boat arrives at the other side. The water speed at time \( t \) is \( f(x) = f(2t) = \frac{2(10 - 2t)}{5} \), so the velocity vector is \( \mathbf{v}(t) = \langle 2, \frac{2(10 - 2t)}{5} \rangle = \langle 2, 4t - \frac{4}{5}t^2 \rangle \). The position vector at time \( t = 5 \) is \( \mathbf{r}(5) = \int_{0}^{5} \langle 2, 4t - \frac{4}{5}t^2 \rangle dt = \langle 2t, 2t^2 - \frac{4}{15}t^3 \rangle \bigg|_{0}^{5} = \langle 10, 50 - \frac{100}{3} \rangle = \langle 10, \frac{50}{3} \rangle \). So the answer is \( 50/3 = 16.67 \) meters.

8. (15 pts) A wagon is pulled a distance of 50 m by a constant force of 20 N. The handle of the wagon is held at an angle of 45°. How much work is done?

\[
W = |\mathbf{F}| |\mathbf{D}| \cos \alpha = 20 \cdot 50 \cos 45° = \frac{1000}{\sqrt{2}}
\]

9. (15 pts) Find the length of the curve \( \mathbf{r}(t) = \langle 3t + 1, 4t^{3/2} - 1, 3t^2 \rangle \), \( 0 \leq t \leq 1 \).

\[
\mathbf{r}'(t) = \langle 3, 6t^{1/2}, 6t \rangle,
\]

so \( L = \int_{0}^{1} |\mathbf{r}'(t)| dt = \int_{0}^{1} \sqrt{9 + 36t + 36t^2} dt = 3 \int_{0}^{1} \sqrt{1 + 4t + 4t^2} dt = 3 \int_{0}^{1} (1 + 2t) dt = 3 \left[ t + t^2 \right]_{0}^{1} = 6. \)