**Homework 1 Solutions**

1.1.4 (a) Prove that $A \subseteq B$ iff $A \cap B = A$.

*Proof.* First assume that $A \subseteq B$. If $x \in A \cap B$, then $x \in A$ and $x \in B$ by definition, so in particular $x \in A$. This proves $A \cap B \subseteq A$. Now if $x \in A$, then by assumption $x \in B$, too, so $x \in A \cap B$. This proves $A \subseteq A \cap B$. Together this implies $A = A \cap B$.

Now assume that $A \cap B = A$. If $x \in A$, then by assumption $x \in A \cap B$, so $x \in A$ and $x \in B$. In particular, $x \in B$. This proves $A \subseteq B$. \qed

1.1.4 (b) Prove $A \cap B = A \setminus (A \setminus B)$.

*Proof.* Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. In particular, $x \notin A \setminus B$ (because $x \in A \setminus B$ would imply $x \notin B$). So $x \in A \setminus (A \setminus B)$. This shows $A \cap B \subseteq A \setminus (A \setminus B)$. Now let $x \in A \setminus (A \setminus B)$. Then $x \in A$ and $x \notin A \setminus B$. This means that $x \notin A$ or $x \in B$ (the negation of $x \in A$ and $x \notin B$). Since we know $x \in A$, this implies $x \in B$, so $x \in A \cap B$. This shows $A \setminus (A \setminus B) \subseteq A \cap B$. Together with the first part this shows $A \cap B = A \setminus (A \setminus B)$. \qed

1.1.4 (c) Prove $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

*Proof.* Let $x \in (A \setminus B) \cup (B \setminus A)$. Then $x \in A \setminus B$ or $x \in B \setminus A$. In the first case, this implies $x \in A$ and $x \notin B$. From this we get $x \in A$ or $x \in B$ (since the first of those statements is true), so $x \in A \cup B$. We also get that $x \notin A \cap B$ (because $x \notin B$), so $x \in (A \cup B) \setminus (A \cap B)$. In the second case we get $x \in B$ and $x \notin A$, so by the same argument $x \in A \cup B$ and $x \notin A \cap B$. Again we conclude $x \in (A \cup B) \setminus (A \cap B)$. This shows $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$.

Now let $x \in (A \cup B) \setminus (A \cap B)$. Then $x \in A$ or $x \in B$, and $x \notin A \cap B$. If $x \in A$, then $x \notin B$ (because otherwise $x \in A \cap B$), so $x \in A \setminus B$. If $x \notin A$, then by assumption $x \in B$, so $x \in B \setminus A$. In either case, $x \in (A \setminus B) \cup (B \setminus A)$. This shows $(A \cup B) \setminus (A \cap B) \subseteq x \in (A \setminus B) \cup (B \setminus A)$. Together with the first part this shows the claimed set equality. \qed

1.1.4 (d) Prove that $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

*Proof.* If $p \in (A \cap B) \times C$, then $p = (x, y)$ with $x \in A \cap B$ and $y \in C$. This means $x \in A$, $x \in B$ and $y \in C$, and thus $(x, y) \in A \times C$ and $(x, y) \in B \times C$. This implies $p = (x, y) \in (A \times C) \cap (B \times C)$. This proves $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$.

If $p \in (A \times C) \cap (B \times C)$, then $p \in A \times C$ and $p \in B \times C$, so $p = (x, y)$ with $x \in A$ and $y \in C$, and $x \in B$ and $y \in C$. This implies $x \in A \cap B$ and $y \in C$, so $p = (x, y) \in (A \cap B) \times C$. This proves $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$. Together the two inclusions prove the claimed equality. \qed
1.1.4 (e) Prove that \( A \cap B \) and \( A \setminus B \) are disjoint, and that \( A = (A \cap B) \cup (A \setminus B) \).

**Proof.** For the first part we have to prove that \((A \cap B) \cap (A \setminus B) = \emptyset\). Let \( x \in (A \cap B) \cap (A \setminus B) \). Then \( x \in A \cap B \) and \( x \in A \setminus B \), so \( x \in A \) and \( x \in B \), and \( x \in A \) and \( x \notin B \). In particular, this implies \( x \in B \) and \( x \notin B \), which is a contradiction. I.e., there can be no such \( x \in A \). We proved that \((A \cap B) \cap (A \setminus B) = \emptyset\).

For the set equality, let \( x \in A \) be arbitrary. Then either \( x \in B \) or \( x \notin B \). In the first case, \( x \in A \cap B \), in the second case \( x \in A \setminus B \). In either case, \( x \in (A \cap B) \cup (A \setminus B) \). This shows \( A \subseteq (A \cap B) \cup (A \setminus B) \).

Now let \( x \in (A \cap B) \cup (A \setminus B) \). Then \( x \in A \cap B \) or \( x \in A \setminus B \). Either case implies \( x \in A \) by definition. This shows \((A \cap B) \cup (A \setminus B) \subseteq A \). Together the two inclusions show the claimed set equality. \(\square\)

1.2.5 Prove that if a function \( f \) has a maximum, then \( \sup f \) exists and \( \max f = \sup f \).

**Proof.** For the existence of the supremum we have to show that \( f \) is bounded above, and for the claimed equality we have to show that \( \max f \) is the least upper bound for \( f \).

By definition of the maximum, there exists \( x_0 \in X \) with \( f(x) \leq f(x_0) = \max f \) for all \( x \in X \). This shows that \( \max f \) is an upper bound for \( f \), and that the supremum of \( f \) exists.

Now choose an arbitrary \( M \in \mathbb{R} \) with \( M < \max f \). Then \( M < f(x_0) \), and thus \( M \) is not an upper bound. This shows that \( \max f \) is the least upper bound, i.e., \( \max f = \sup f \). \(\square\)

1.2.22 Suppose that \( f : X \rightarrow Y \).

For the following proofs we break down the “if and only if” into both directions. The symbol “\( \implies \)” means that we show that the first assumption implies the second one, the symbol “\( \iff \)” means that we are proving that the second assumption implies the first one. Similarly we break down the proof of set equalities into the two inclusions “\( \subseteq \)” and “\( \supseteq \)”.

1.2.22 (a) Prove that \( f(A \cap B) = f(A) \cap f(B) \) for all \( A, B \subseteq X \) if \( f \) is injective.

**Proof.** We show the implications separately.

\( \implies \): Let \( x_1, x_2 \in X \) be arbitrary with \( f(x_1) = f(x_2) \). Let \( A = \{x_1\} \) and \( B = \{x_2\} \). By assumption, \( f(A \cap B) = f(A) \cap f(B) = \{f(x_1)\} \cap \{f(x_2)\} = \{f(x_1)\} \). This implies that there exists an element \( x \in A \cap B \) with \( f(x) = f(x_1) \). Since \( x \in A \) and \( x \in B \) we have that \( x = x_1 \) and \( x = x_2 \), and hence \( x_1 = x_2 \). This shows that \( f \) is injective.
Proof. This breaks down into two parts itself.

\(\subseteq\): Let \(y \in f(A \cap B)\). Then there exists \(x \in A \cap B\) with \(f(x) = y\). This implies that \(x \in A\) and \(x \in B\) with \(f(x) = y\), thus \(y \in f(A)\) and \(y \in f(B)\). By definition, \(y \in f(A) \cap f(B)\).

\(\supseteq\): Let \(y \in f(A) \cap f(B)\). Then \(y \in f(A)\) and \(y \in f(B)\). Thus there exists \(x_1 \in A\) with \(f(x_1) = y\) and there exists \(x_2 \in B\) with \(f(x_2) = y\). By injectivity of \(f\) we have \(x_1 = x_2\), and thus \(x_1 \in B\), too. So \(x_1 \in A \cap B\) and hence \(y = f(x_1) \in f(A \cap B)\). \(\square\)

1.2.22 (b) Prove that \(f(A \setminus B) = f(A) \setminus f(B)\) for all \(A, B \subseteq X\) iff \(f\) is injective.

Proof. Set difference is intersection with the complement, so this proof mimicks the proof in (a).

\(\implies\): Let \(x_1, x_2 \in X\) be arbitrary with \(f(x_1) = f(x_2)\). Let \(A = \{x_1\}\) and \(B = \{x_2\}\). By assumption, \(f(A \setminus B) = f(A) \setminus f(B) = \{f(x_1)\} \setminus \{f(x_2)\} = \emptyset\). This implies that \(A \setminus B = \emptyset\), and hence \(\{x_1\} \setminus \{x_2\} = \emptyset\). This means that \(x_1 = x_2\) (because otherwise \(\{x_1\} \setminus \{x_2\} = \{x_1\}\)). This shows that \(f\) is injective.

\(\impliedby\): This breaks down into two parts itself.

\(\subseteq\): Let \(y \in f(A \setminus B)\). Then there exists \(x \in A \setminus B\) with \(f(x) = y\). This implies that \(x \in A\) and \(x \notin B\) with \(f(x) = y\). We can immediately deduce \(y \in f(A)\). Now we have to show that \(y \notin f(B)\). Assume to the contrary that \(y \in f(B)\). Then there exists \(x_1 \in B\) with \(f(x_1) = y\). By injectivity of \(f\), we get \(x = x_1\), and thus \(x \in B\) and \(x \notin B\), a contradiction. This shows that \(y \notin f(B)\), and thus \(y \in f(A) \setminus f(B)\).

\(\supseteq\): Let \(y \in f(A) \setminus f(B)\). Then \(y \in f(A)\) and \(y \notin f(B)\). Thus there exists \(x \in A\) with \(f(x) = y\). If \(x \in B\), then \(y \in f(B)\), which contradicts the previous statement, so we must have \(x \notin B\). This implies \(x \in A \setminus B\), and hence \(y \in f(A \setminus B)\). \(\square\)

1.2.22 (c) Prove that \(f^{-1}(f(A)) = A\) for all \(A \subseteq X\) iff \(f\) is injective.

Proof. \(\implies\): Let \(x_1, x_2 \in X\) with \(f(x_1) = f(x_2)\). Let \(A = \{x_1\}\). Then \(f(A) = \{f(x_1)\}\), and since \(f(x_1) = f(x_2)\) we have that \(x_2 \in f^{-1}(f(A))\). By assumption \(f^{-1}(f(A)) = A\), so \(x_2 \in A = \{x_1\}\), and thus \(x_1 = x_2\). This shows that \(f\) is injective.

\(\impliedby\):

\(\subseteq\): Let \(x \in f^{-1}(f(A))\). Then \(f(x) \in f(A)\), hence there exists \(x_1 \in A\) with \(f(x_1) = f(x)\). By injectivity, \(x = x_1\), so \(x \in A\).

\(\supseteq\): Let \(x \in A\). Then \(f(x) \in f(A)\), and by definition this implies \(x \in f^{-1}(f(A))\). \(\square\)
1.2.22 (d) Prove that $f(f^{-1}(B)) = B$ for all $B \subseteq Y$ iff $f$ is surjective.

Proof. $\implies$: Let $y \in Y$ arbitrary. We have to show that there exists $x \in X$ with $f(x) = y$. Let $B = \{y\}$. By assumption, $f(f^{-1}(B)) = B = \{y\}$, so $y \in f(f^{-1}(B))$. By definition this means that there exists $x \in f^{-1}(B)$ with $f(x) = y$.

$\impliedby$: Let $y \in f(f^{-1}(B))$. Then there exists $x \in f^{-1}(B)$ with $f(x) = y$. By surjectivity of $f$ there exists $x \in X$ with $f(x) = y$. This implies that $x \in f^{-1}(B)$.

For problems 23 and 24 we will choose $X = Y = \mathbb{R}$ and the functions $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. (Since $f$ is neither injective nor surjective it is a good candidate for counterexamples.)

1.2.23 (a) Find an example for which $f^{-1}(f(A)) \neq A$.

$A = \{1\}$ gives $f(A) = \{1\}$ and $f^{-1}(f(A)) = \{-1, 1\} \neq A$.

1.2.23 (b) Find an example for which $f(f^{-1}(A)) \neq A$.

$A = \{-1\}$ gives $f^{-1}(A) = \emptyset$ and $f(f^{-1}(A)) = \emptyset \neq A$.

1.2.24 (a) Find an example for which $f(A \cap B) \neq f(A) \cap f(B)$.

$A = \{1\}$ and $B = \{-1\}$ give $A \cap B = \emptyset$, $f(A \cap B) = \emptyset$, $f(A) = \{1\}$ and $f(B) = \{1\}$, so $f(A) \cap f(B) = \emptyset \neq f(A \cap B)$.

1.2.24 (b) Find an example for which $f(A \setminus B) \neq f(A) \setminus f(B)$.

$A = \{1\}$ and $B = \{-1\}$ give $A \setminus B = \{1\}$, $f(A \setminus B) = \{1\}$, $f(A) = f(B) = \{1\}$, and $f(A) \setminus f(B) = \emptyset \neq f(A \setminus B)$.

As you can see, we could as well have chosen $X = Y = \{-1, 1\}$ and $f : X \to Y$ given by $f(1) = f(-1) = 1$ in all counterexamples.
1.4.1 (a) The negation of “there exists \( p > 0 \) such that for every \( x \) we have \( f(x + p) = f(x) \)” is “for all \( p > 0 \) there exists \( x \) with \( f(x + p) \neq f(x) \).”

In formal notation with quantifiers (using \( \equiv \) for logical equivalence):

\[
\sim (\exists p > 0 \forall x : f(x + p) = f(x)) \equiv \forall p > 0 \exists x : f(x + p) \neq f(x)
\]

Mathematically, the original statement means that \( f \) is a periodic function.

1.4.1 (b) The negation of “for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( x \) and \( t \) are in \( D \) and satisfy \( |x - t| < \delta \), then \( |f(x) - f(t)| < \epsilon \)” is “there exists \( \epsilon > 0 \) such that for every \( \delta > 0 \) there exist \( x \) and \( t \) in \( D \) with \( |x - t| < \delta \), but \( |f(x) - f(t)| \geq \epsilon \).”

Again the same in formal notation:

\[
\sim (\forall \epsilon > 0 \exists \delta > 0 \forall x, t \in D : |x - t| < \delta \Rightarrow |f(x) - f(t)| < \epsilon) \\
\equiv \exists \epsilon > 0 \forall \delta > 0 \exists x, t \in D : (|x - t| < \delta) \land (|f(x) - f(t)| \geq \epsilon)
\]

Mathematically, the original statement means that the function \( f \) is uniformly continuous in \( D \).

1.4.1 (c) The negation of “for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( x \in D \) and \( 0 < |x - a| < \delta \), then \( |f(x) - A| < \epsilon \)” is “there exists \( \epsilon > 0 \) such that for all \( \delta > 0 \) there exists \( x \in D \) with \( 0 < |x - a| < \delta \), but \( |f(x) - A| \geq \epsilon \).”

In formal notation:

\[
\sim (\forall \epsilon > 0 \exists \delta > 0 \forall x \in D : 0 < |x - a| < \delta \Rightarrow |f(x) - A| < \epsilon) \\
\equiv \exists \epsilon > 0 \forall \delta > 0 \exists x \in D : (0 < |x - a| < \delta) \land (|f(x) - A| \geq \epsilon)
\]

Mathematically the original statement means that \( \lim_{x \to a} f(x) = A \) (assuming that the domain of the function is \( D \)).

1.4.5 Consider the statement \( P \): the sum of two irrational numbers is irrational.

1.4.5 (a) Give an example of a case in which \( P \) is true.

\[ \sqrt{2} + \sqrt{2} = 2\sqrt{2} \]. (To show that \( 2\sqrt{2} \) is irrational, assume to the contrary that \( 2\sqrt{2} = p/q \) with integers \( p \) and \( q \). Then \( \sqrt{2} = p/(2q) \) would be rational as well. But we proved in class that this is not true, so this is a contradiction, and thus \( 2\sqrt{2} \) is irrational.)

1.4.5 (b) Prove or disprove \( P \) by giving a counterexample.

The statement is not true in general: \( \sqrt{2} \) and \( -\sqrt{2} \) are irrational, but their sum \( \sqrt{2} + (-\sqrt{2}) = 0 \) is rational. (If you want an example with positive numbers, choose \( \sqrt{2} \) and \( 2 - \sqrt{2} \). Irrationality of \( 2 - \sqrt{2} \) follows in the same way as that of \( 2\sqrt{2} \).)