1.3.2 (d) Prove $\sum_{k=1}^{n} (2k - 1) = n^2$ by mathematical induction.

Proof. For $n = 1$ we have $\sum_{k=1}^{1} (2k - 1) = 2 \cdot 1 - 1 = 1 = 1^2$. Assuming now that the statement is true for $n$, we get

$$\sum_{k=1}^{n+1} (2k - 1) = \sum_{k=1}^{n} (2k - 1) + 2(n + 1) - 1 = n^2 + 2n + 1 = (n + 1)^2.$$  

$\square$

1.3.2 (s) Prove $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$ for all $n \geq 2$ by mathematical induction.

Proof. For $n = 2$ we have $\sum_{k=1}^{2} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} = \frac{\sqrt{2} + 1}{\sqrt{2}} > \frac{2}{\sqrt{2}} = \sqrt{2}$. Assuming that the statement is true for $n$, we get

$$\sum_{k=1}^{n+1} \frac{1}{\sqrt{k}} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{n + 1}} > \sqrt{n} + \frac{1}{\sqrt{n + 1}} = \frac{\sqrt{n} \sqrt{n + 1} + 1}{\sqrt{n + 1}}$$

$$> \frac{\sqrt{n} \sqrt{n + 1} + 1}{\sqrt{n + 1}} = \frac{n + 1}{\sqrt{n + 1}} = \sqrt{n + 1}.$$  

$\square$

1.3.6 (c) Use mathematical induction to prove the following: If $a_1 = 1$ and $a_{n+1} = \sqrt{3a_n + 1}$ for all $n \in \mathbb{N}$, then $a_n < \frac{7}{2}$ for all $n \in \mathbb{N}$.

Proof. $a_2 = \sqrt{3a_1 + 1} = \sqrt{3 + 1} = 2$, so $a_1 = 1 < 2 = a_2$. Assuming that we already know $a_n < a_{n+1}$, we know that $3a_n + 1 < 3a_{n+1} + 1$, so $a_{n+1} = \sqrt{3a_n + 1} < \sqrt{3a_{n+1} + 1} = a_{n+2}$.  

$\square$

1.3.6 (d) Use mathematical induction to prove the following: If $a_1 = 1$ and $a_{n+1} = \sqrt{3a_n + 1}$ for all $n \in \mathbb{N}$, then $a_n < \frac{7}{2}$ for all $n \in \mathbb{N}$.

Proof. $a_1 = 1 < \frac{7}{2}$ is immediate. Now assume that we already know $a_n < \frac{7}{2}$. Then $3a_n + 1 < 3 \cdot \frac{7}{2} + 1 = \frac{23}{2} < \frac{49}{4}$, so $a_{n+1} = \sqrt{3a_n + 1} < \sqrt{\frac{49}{4}} = \frac{7}{2}$.  

$\square$
1.5.12 Suppose that \( f : A \to B \) and \( g : B \to C \) are two bijections. Prove that \( g \circ f : A \to C \) has an inverse function \( f^{-1} \circ g^{-1} : C \to A \). (This verifies \( (g \circ f)^{-1} = f^{-1} \circ g^{-1} \).)

**Proof.** We use Theorem 1.5.6 (c) to check that \( g \circ f \) is a bijection with inverse \( f^{-1} \circ g^{-1} \). We know that 

\[
(f^{-1} \circ g^{-1}) \circ (g \circ f)(x) = f^{-1}(g^{-1}(g(f(x)))) = f^{-1}(f(x)) = x
\]

for all \( x \in A \), and 

\[
(g \circ f) \circ (f^{-1} \circ g^{-1})(z) = g(f(f^{-1}(g^{-1}(z)))) = g(g^{-1}(z)) = z
\]

for all \( z \in C \). By Theorem 1.5.6 (c) this implies that \( g \circ f \) is bijective with \( (g \circ f)^{-1} = f^{-1} \circ g^{-1} \).

(We used the facts that \( f^{-1}(f(x)) = x \) for all \( x \in A \); \( f(f^{-1}(y)) = y \) for all \( y \in B \); \( g^{-1}(g(y)) = y \) for all \( y \in B \); and \( g(g^{-1}(z)) = z \) for all \( z \in C \). All of these directly follow from Theorem 1.5.4.)

\[ \Box \]

1.5.15 (a) \( \arctan \left( \tan \frac{3\pi}{4} \right) = -\frac{\pi}{4} \).

1.5.15 (b) \( \arctan(\tan x) = x \) for \( -\frac{\pi}{2} < x < \frac{\pi}{2} \).

1.5.15 (c) \( \tan(\arctan x) = x \) for all \( x \in \mathbb{R} \).