Extra Problem. Prove that \((0, 1), (0, 1], [0, 1], \) and \(\mathbb{R}\) are equivalent sets.

**Proof.** The easiest equivalence is \((0, 1) \sim \mathbb{R}\), one possible bijection is given by \(f : (0, 1) \to \mathbb{R}\),
\[
f(x) = \begin{cases} 
2 - \frac{1}{x} & \text{for } 0 < x < \frac{1}{2}, \\
\frac{1}{1-x} - 2 & \text{for } \frac{1}{2} \leq x < 1, 
\end{cases}
\]
with inverse function
\[
f^{-1}(y) = \begin{cases} 
\frac{1}{2-y} & \text{for } y < 0, \\
1 - \frac{1}{y+2} & \text{for } y \geq 0.
\end{cases}
\]
To show \((0, 1] \sim (0, 1)\), one possible bijection \(g : (0, 1] \to (0, 1)\) is given by
\[
g(x) = \begin{cases} 
\frac{1}{n+1} & \text{for } x = \frac{1}{n}, n \in \mathbb{N}, \\
x & \text{if } x \neq \frac{1}{n} \text{ for all } n \in \mathbb{N},
\end{cases}
\]
with inverse \(g^{-1}(y) = \begin{cases} 
\frac{1}{n-1} & \text{for } y = \frac{1}{n}, n \in \mathbb{N}, n \geq 2, \\
y & \text{if } y \neq \frac{1}{n} \text{ for all } n \in \mathbb{N}, n \geq 2.
\end{cases}\)
Then \(h : [0, 1] \to [0, 1)\) defined by
\[
h(x) = \begin{cases} 
g(x) & \text{for } x \neq 0, \\
0 & \text{if } x = 0,
\end{cases}
\]
is again a bijection, so \([0, 1] \sim [0, 1)\). But \(F : [0, 1) \to (0, 1], F(x) = 1-x\) is a bijection, too (with \(F^{-1} = F\)), so \([0, 1) \sim (0, 1]\). By transitivity the claimed equivalences follow. \(\square\)

1.8.5. (d) Find all real values \(x\) such that
\[
\frac{2x + 1}{x - 5} \leq 3.
\]
We consider the cases \(x \geq 5\) and \(x < 5\) separately. If \(x \geq 5\), the inequality becomes \(2x + 1 \leq 3(x - 5)\), which is equivalent to \(x \geq 16\). If \(x < 5\), then we get \(2x + 1 \geq 3(x - 5)\), leading to \(x \leq 16\). From the first case we get all \(x \in [16, \infty)\), from the second case we get all \(x \in (-\infty, 5)\), so the set of all possible solutions is \([16, \infty) \cup (-\infty, 5)\).

1.8.13. If \(a, b \in \mathbb{R}\) and \(a - \epsilon < b\) for any \(\epsilon > 0\), prove that \(a \leq b\).
Proof. Assume not, i.e., $a > b$. Let $\epsilon = a - b > 0$. Then by assumption $a - \epsilon < b$, so $b = a - (a - b) = < b$. This is a contradiction. \qed

1.8.14. (c) Prove that $|a| = \sqrt{a^2}$.

Proof. The square root of a number $y \geq 0$ is defined as the number $x \geq 0$ such that $x^2 = y$. So we have to show that $|a| \geq 0$ and that $|a|^2 = a^2$. If $a \geq 0$, we have $|a| = a \geq 0$ and $|a|^2 = a^2$. If $a < 0$, then we get $|a| = a > 0$ and $|a|^2 = (-a)^2 = a^2$. Thus we get the claim in both cases. \qed

1.8.15. (b) Find all real values of $x$ that satisfy $|2x - 5| \leq |x + 4|$.

We have to consider different cases depending on the signs of the expressions between absolute value signs.

If $x \geq 5/2$, then $2x - 5 \geq 0$ and $x + 4 \geq 0$, so the inequality is $2x - 5 \leq x + 4$, equivalent to $x \leq 9$. This yields the interval $[5/2, 9]$ as part of the solution.

If $-4 \leq x < 5/2$, then $2x - 5 < 0$ and $x + 4 \geq 0$, so the inequality is $-(2x - 5) \leq x + 4$, equivalent to $x \geq 1/3$. This gives $[1/3, 5/2)$ as part of the solution.

If $x < -4$, then $2x - 5 < 0$, and $x + 4 < 0$, so the inequality is $-(2x - 5) \leq -(x + 4)$, equivalent to $x \geq 9$. However, there are no $x$ which satisfy both $x < -4$ and $x \geq 9$, so this does not give any more solutions.

Putting everything together, the set of values of $x$ for which the inequality is satisfied is $[5/2, 9] \cup [1/3, 5/2) = [1/3, 9]$.

1.8.15. (c) Find all real values of $x$ such that $2|1 - 3x| > 5$.

This is equivalent to $|1 - 3x| > 5/2$, and this is satisfied if and only if $1 - 3x > 5/2$ or $1 - 3x < -5/2$. This is equivalent to $x < -1/2$ or $x > 7/6$, so the set of solutions is $(-\infty, -1/2) \cup (7/6, \infty)$.

1.8.15. (f) Find all real values of $x$ such that $|\frac{2x + 1}{x - 3}| \geq -1$.

Absolute values are always non-negative, so they are certainly greater than $-1$. This means that this is always true whenever it is well-defined, i.e., for all $x \in \mathbb{R}$, $x \neq 3$.

1.8.15. (g) Find all real values of $x$ such that $|x + 6| = |2x + 1|$.

This true if and only if $x + 6 = 2x + 1$ or $x + 6 = -(2x + 1)$. This means that the only solutions are $x = 5$ and $x = -5/3$. 