3.1.2 (a) \( \lim_{x \to \infty} f(x) = 2 \).

Proof. Let \( \epsilon > 0 \). Then for \( x \in D \) with \( x > 3 + 6/\epsilon \) we have \( x - 3 > 6/\epsilon > 0 \) and thus

\[
|f(x) - 2| = \left| \frac{2x}{x-3} - 2 \right| = \left| \frac{6}{x-3} \right| = \frac{6}{x-3} < \frac{6}{6/\epsilon} = \epsilon.
\]

\( \square \)

3.1.2 (b) \( \lim_{x \to \infty} \frac{1-x^2}{x-2} = -\infty \).

Proof. Let \( K > 0 \). For \( x > 2 \) we have \( x^2 - 1 > \frac{x^2}{x} \) and \( x - 2 > x > 0 \), so \( \frac{x^2-1}{x-2} > \frac{x^2/2}{x} = x/2 \). So for \( x > \max(2, 2K) \) we get

\[
\frac{1-x^2}{x-2} < -\frac{x}{2} < -K.
\]

\( \square \)

3.1.2 (c) \( \lim_{x \to -\infty} f(x) = -\infty \).

Proof. Let \( K > 0 \). For \( x < -2 \) we have \( x^2 + 1 > x^2 > 0 \) and \( 0 < 2-x < -2x \), so \( \frac{x^2+1}{2-x} > \frac{x^2}{2x} = -x/2 \). So for \( x \in \mathbb{Q} \) with \( x < -\max(2, 2K) \) we get

\[
\frac{x^2 + 1}{x - 2} < \frac{x}{2} < -K.
\]

\( \square \)

3.1.2 (d) \( \lim_{x \to -\infty} \frac{-1}{x+1} = 0 \).

Proof. Let \( \epsilon > 0 \). Then for \( x < -1 - 1/\epsilon \) we have \( x + 1 < -1/\epsilon < 0 \) and thus \( |x+1| > 1/\epsilon > 0 \). This implies \( |f(x) - 0| = \left| \frac{-1}{x+1} \right| = \frac{1}{|x+1|} < \epsilon \). \( \square \)

3.1.5 (c) \( \lim_{n \to \infty} \frac{-2n}{3\sqrt{n^2-1}} = -2/3 \).

Proof. \( \frac{-2n}{3\sqrt{n^2-1}} = \frac{-2}{3\sqrt{1-1/n^2}} \) and we know \( \lim_{n \to \infty} 1/n^2 = 0 \). Applying limit theorems yields the result. \( \square \)
3.1.5 (g) \( \lim_{x \to -\infty} \sqrt{x} = \infty \).

**Proof.** Let \( K > 0 \). Then for \( x > K^2 \) we have \( \sqrt{x} > K \). \( \square \)

3.1.5 (i) \( \lim_{x \to -\infty} \frac{x - 3}{|x - 3|} = -1 \).

**Proof.** For \( x < 3 \) we have \( |x - 3| = -(x - 3) \), so \( \frac{x - 3}{|x - 3|} = -1 \). In particular, \( \left| \frac{x - 3}{|x - 3|} - (-1) \right| = 0 < \epsilon \) for any \( \epsilon > 0 \). \( \square \)

3.1.6 (a) E.g., the function

\[
 f(x) = \begin{cases} 
 \frac{1}{x} & \text{for } x \neq 0, \\
 0 & \text{for } x = 0, 
\end{cases}
\]

is unbounded on \( \mathbb{R} \), yet \( \lim_{x \to -\infty} f(x) = 0 \) is finite.

3.1.6 (b) If \( \lim_{x \to \infty} f(x) = L \in \mathbb{R} \), then for every \( \epsilon > 0 \) there exists \( M > 0 \) such that \( |f(x) - L| < \epsilon \) whenever \( x \in D \) with \( x \geq M \). Now if \( t \) is a number with \( -t \in D \) and \( t \leq -M \), then \( -t \geq M \), and thus \( |f(-t) - L| < \epsilon \). This shows that \( \lim_{t \to -\infty} f(-t) = L \).

The other direction works exactly the same way, and the cases where \( L = \pm \infty \) are simple modifications.

3.1.6 (c) \( \lim_{x \to -\infty} 2^x e^{-x} = \lim_{x \to \infty} 2^{-x} e^x = \lim_{x \to -\infty} (\epsilon/2)^x = \infty \) since \( \epsilon/2 > 1 \).

3.2.1 (a) \( \lim_{x \to -0} (x + 1)^3 = 1 \).

**Proof.** Let \( \epsilon > 0 \). Then for \( |x| < \delta := \min(1, \epsilon/7) \) we get \( |x^2 + 3x + 3| \leq |x|^2 + 3|x| + 3 < 1 + 3 + 3 = 7 \), because \( |x| < 1 \). This implies \( |(x + 1)^3 - 1| = |x^3 + 3x^2 + 3x| = |x||x^2 + 3x + 3| \leq 7|x| < \epsilon \), since \( |x| < \epsilon/7 \). \( \square \)

3.2.1 (d) \( \lim_{x \to 0} \frac{x^2}{|x|} = 0 \).

**Proof.** Let \( \epsilon > 0 \). Then for \( 0 < |x| < \delta := \epsilon \) we get \( \left| \frac{x^2}{|x|} - 0 \right| = |x| < \delta = \epsilon \). \( \square \)
3.2.1 (f) \( \lim_{x \to 1} \frac{1-x}{1-\sqrt{x}} = 2. \)

Proof. Let \( \epsilon > 0 \), and choose \( \delta = \epsilon \). Let \( x \geq 0 \) with \( |x-1| < \delta \). We have
\[
\frac{1-x}{1-\sqrt{x}} - 2 = \frac{1 - x - 2 + 2\sqrt{x}}{1 - \sqrt{x}} = \frac{1 + 2\sqrt{x} - x}{1 - \sqrt{x}} = \frac{-(1-\sqrt{x})^2}{1 - \sqrt{x}} = |1 + \sqrt{x}| = \frac{\sqrt{x} - 1}{\sqrt{x} + 1} \leq |x-1| < \epsilon.
\]
\[\square\]

3.2.8 (a) \( \lim_{x \to 3/8} f(x) = 1. \)

Proof. Let \( \epsilon > 0 \), and choose \( \delta = 1/24 \). If \( x \in \mathbb{R} \) with \( |x - 3/8| < 1/24 \), then \( 1/3 = 3/8 - 1/24 < x < 3/8 + 1/24 < 1/2 \), so \( 2 < 1/x < 3 \). In particular, \( x \) can not be the reciprocal of an integer, and thus \( f(x) = 1 \), and \( |f(x) - 1| = 0 < \epsilon \).
\[\square\]

3.2.8 (b) \( \lim_{x \to -1/3} f(x) = 1. \)

Proof. Let \( \epsilon > 0 \), and choose \( \delta = 1/12 \). If \( x \in \mathbb{R} \) with \( 0 < |x - (-1/3)| < 1/12 \), then \( -1/2 < -1/3 - 1/12 < x < -1/3 + 1/12 = -1/4 \), so \( -4 < 1/x < -2 \). In particular, the only way that \( x \) can be the reciprocal of an integer is \( x = -1/3 \). However, this contradicts \( 0 < |x - (-1/3)| \), and thus \( f(x) = 1 \), and \( |f(x) - 1| = 0 < \epsilon \).
\[\square\]

3.2.8 (c) \( \lim_{x \to 0} f(x) \) does not exist.

Proof. Let \( x_n = 1/n \). Then \( \lim_{n \to \infty} x_n = 0 \) and \( \lim_{n \to \infty} f(x_n) = 0 \). Let \( y_n = \sqrt{n} \). Then \( y_n \) is irrational for all \( n \), and hence not the reciprocal of an integer. This implies \( f(y_n) = 1 \), and thus \( \lim_{n \to \infty} y_n = 0 \) and \( \lim_{n \to \infty} f(y_n) = 1 \).
Since the limits of \( \{f(x_n)\} \) and \( \{f(y_n)\} \) are different, the limit of \( f(x) \) as \( x \) tends to 0 does not exist.
\[\square\]

Squeeze Theorem If \( f, g, h : D \to \mathbb{R} \) are functions with \( \lim_{x \to \infty} f(x) = A = \lim_{x \to \infty} h(x) \), and \( f(x) \leq g(x) \leq h(x) \) eventually, then \( \lim_{x \to \infty} g(x) = A \).