1. True or false?

(a) If $g \circ f$ is one-to-one, then $f$ is one-to-one.

True. If $f(x) = f(y)$, then $g(f(x)) = g(f(y))$, and by assumption this implies $x = y$.

(b) If $g \circ f$ is one-to-one, then $g$ is one-to-one.

False. Counterexample: $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$ is not one-to-one, but with $f : (0, +\infty) \to \mathbb{R}$, $f(x) = x$, the composition $g \circ f : (0, +\infty) \to \mathbb{R}$, $(g \circ f)(x) = x^2$ is one-to-one.

(c) If $(a_n)$ is a bounded sequence, then $\sum a_n$ converges.

False. Counterexamples for this are slightly tricky to write down, one possibility is the following. Let $(a_n)$ be a sequence of numbers $\pm 1$ constructed inductively as follows: Let $a_1 = -1$. Now assume that $a_1, \ldots, a_n$ have already been determined. Define $a_{n+1} = \ldots = a_{2n} = 1$, and $a_{2n+1} = \ldots = a_{5n} = -1$. Then $b_{2n} \geq 0$ and $b_{5n} \leq -1/5$. Since $|b_{2n} - b_{5n}| \geq 1/5$, this shows that $(b_n)$ is not a Cauchy sequence, hence divergent.

(d) If $b_n = \sum a_n$ converges, then $(a_n)$ is bounded.

False. One possible counterexample is the unbounded sequence $a_{2k-1} = \sqrt{k}$, $a_{2k} = -\sqrt{k}$. Then $b_{2k} = 0$, and $b_{2k-1} = \frac{\sqrt{k}}{2k-1}$, so $\lim_{k \to \infty} b_{2k} = \lim_{k \to \infty} b_{2k-1} = 0$. This shows that $\lim_{n \to \infty} b_n = 0$.

(e) Every continuous function $f : [0, +\infty) \to \mathbb{R}$ is bounded.

False. Counterexample $f(x) = x$.

(f) Every continuous function $f : [0, +\infty) \to \mathbb{R}$ with $\lim_{x \to \infty} f(x) = 0$ is bounded.

True. The assumption implies that $f$ is eventually bounded, i.e., there exists $M$ and $K_1$ such that $|f(x)| \leq K_1$ for $x \geq M$. Continuous functions on closed bounded intervals are bounded, so there exists $K_2$ such that $|f(x)| \leq K_2$ for $x \in [0, M]$. Then $|f(x)| \leq K = \max(K_1, K_2)$ for $x \in [0, +\infty)$.

2. Find the limit of these sequences or show that it does not exist.

(a) $\lim_{n \to \infty} \frac{n}{\sqrt{n+1}} = \infty$

(b) $\lim_{n \to \infty} \frac{3^n - (-2)^n}{3^n + (-2)^n} = 1$.

(c) $\lim_{n \to \infty} \frac{2^n + (-3)^n}{2^n - 3^n}$ does not exist, there are two subsequences converging to 1 and -1, respectively. (The sequence with the typo on the sheet I handed out converges to -1.)

(d) $d_1 = 0$, and $d_{n+1} = d_n^2 + 1/4$ for $n \geq 1$. 


\[ \lim_{n \to \infty} d_n = 1/2. \] The sequence satisfies \( 0 \leq d_n \leq 1/2 \) for all \( n \), and it is increasing. As a monotone and bounded sequence it converges to a limit \( D \in [0, 1/2] \) satisfying \( D = D^2 + 1/4 \). The only solution is \( D = 1/2 \).

(e) \( e_1 = 1 \), and \( e_{n+1} = e_n^2 + 1/4 \) for \( n \geq 1 \).

\[ \lim_{n \to \infty} e_n = +\infty. \] This sequence satisfies \( e_n \geq 1 \) for all \( n \), and it is increasing. A limit \( E \) would have to satisfy \( E = E^2 + 1/4 \) and \( E \geq 1 \). Since there is no such real number, the sequence diverges to \( +\infty \).

3. Find the limits or show that they do not exist.

(a) \( \lim_{x \to \infty} \frac{1+x^2}{x^2-x^3} = 0 \)

(b) \( \lim_{x \to 1} \frac{1+x^2}{x^2-x^3} \) does not exist. The limit from the left and from the right are \(-\infty\) and \(\infty\), respectively.

(c) \( \lim_{x \to 0} \frac{1+x^2}{x^2-x^3} = -\infty \).

4. Where are the following functions continuous?

(a) \( f(x) = [x] \) is continuous in \( \mathbb{R} \setminus \mathbb{Z} \).

(b) \( g(x) = x \) for \( x \in \mathbb{Q} \), and \( g(x) = 1/x \) for \( x \notin \mathbb{Q} \), is continuous in \( \pm 1 \).

5. (a) Show that the equation \( r^x + x = 0 \) has exactly one real solution \( x \) for every \( r > 0 \).

First of all, the function \( f_r(x) = r^x + x \) is strictly increasing, so there can be at most one solution. In order to show existence, we observe that \( \lim_{x \to \infty} f_r(x) = +\infty \) and \( \lim_{x \to -\infty} f_r(x) = -\infty \). So there exist \( a, b \in \mathbb{R} \) with \( f_r(a) < 0 \) and \( f_r(b) > 0 \). The intermediate value theorem shows that there exists \( x(r) \) between \( a \) and \( b \) with \( f_r(x(r)) = 0 \).

(b)∗ Denoting this solution by \( x(r) \), show that this is a continuous function of \( r \).

This is a little harder. Assume that \( x(r) \) is not continuous at some point \( r_0 > 0 \). Then there exists \( \epsilon > 0 \) and a sequence \( (r_n) \) of positive numbers converging to \( r_0 \) with \( |x(r_n) - x(r_0)| \geq \epsilon \) for all \( n \). This implies that \( x(r_n) \geq x(r_0) + \epsilon \) for infinitely many \( n \), or that \( x(r_n) \leq x(r_0) - \epsilon \) for infinitely many \( n \). Let us first assume that the first case applies, and let \( x_\epsilon = x(r_0) + \epsilon \). This implies \( 0 = r_n^{x(r_n)} + x(r_n) \geq r_n^{x_\epsilon} + x_\epsilon \) for infinitely many \( n \). Passing to the limit along this subsequence we get \( 0 \geq r_0^{x_\epsilon} + x_\epsilon > r_0^{x_\epsilon} + x_0 = 0 \), a contradiction. The other case leads to a contradiction in the same way.