1. Find the acute angle between two diagonals of a cube.
This is the angle between the vectors \(\mathbf{v} = \langle 1, 1, 1 \rangle\) and \(\mathbf{w} = \langle 1, 1, -1 \rangle\), so 
\[ \alpha = \cos^{-1} \left( \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \right) = \cos^{-1} \left( \frac{1}{3} \right) = 1.23 = 70.5^\circ. \]

2. Find a parametric equation for the line through \((-2, 2, 4)\) which is perpendicular to the plane \(2x - y + 5z = 12\).
\[ \mathbf{r}(t) = \langle -2 + 2t, 2 - t, 4 + 5t \rangle. \]

3. Identify and sketch the surface \(-4x^2 + y^2 - 4z^2 = 4\).
This is a hyperboloid of two sheets.

4. Sketch and find the length of the curve \(\mathbf{r}(t) = \langle 2t^{3/2}, \cos 2t, \sin 2t \rangle, \quad 0 \leq t \leq \pi. \)
(NOTE: The version I handed out contains a typo leading to a harder integral.)
This curve looks somewhat like a deformed part of a helix along the \(x\)-axis. Its length is 
\[ L = \int_0^{\pi} \|\mathbf{r}'(t)\| \, dt = \int_0^{\pi} \sqrt{(3t^{1/2})^2 + (\sin 2t)^2 + (2 \cos 2t)^2} \, dt = \int_0^{\pi} \sqrt{9t + 4} \, dt = \left[ \frac{3}{2} (9t + 4)^{3/2} \right]_0^{\pi/2} = \frac{3}{2} (9(\pi + 4)^{3/2} - 4^{3/2}) = 12.99 \]

5. A particle starts at the origin with initial velocity \(\langle 1, -1, 3 \rangle\). Its acceleration is 
\[ \mathbf{a}(t) = \langle 6t, 12t^2, -6t \rangle. \] Find its position function.
Velocity is 
\[ \mathbf{v}(t) = \langle 1 + 3t^2, -1 + 4t^3, 3 - 3t^3 \rangle, \]
position vector is 
\[ \mathbf{r}(t) = \langle t + t^3, -t + t^4, 3t - t^3 \rangle. \]

6. Find the directions in which the directional derivative of \(f(x, y) = ye^{-xy}\) at the point \((0, 2)\) has the value 1.
\[ \nabla f(x, y) = \langle -y^2 e^{-xy}, e^{-xy} - xy e^{-xy} \rangle, \text{ so } \nabla f(0, 2) = \langle -4, 1 \rangle. \] If \(\mathbf{u}\) is a unit vector which makes an angle of \(\alpha\) with \(\langle -4, 1 \rangle\), then 
\[ D_\mathbf{u} f(0, 2) = \|\nabla f(0, 2)\| \cos \alpha = \sqrt{17} \cos \alpha. \]
This is equal to 1 if \(\alpha = \pm \cos^{-1} \left( \frac{1}{\sqrt{17}} \right) = \pm 1.33. \) The direction of \(\langle -4, 1 \rangle\) is \(\beta = \pi + \tan^{-1} \left( \frac{1}{4} \right) = 2.90, \) so the directions in which the directional derivative is 1 are \(\beta \pm \alpha, \) i.e., 1.57 and 4.22. The corresponding unit vectors are \(\langle 0, 1 \rangle\) and \(\langle -.47, -.88 \rangle. \)

7. Find equations of the tangent plane and the normal line to the surface \(\sin(\pi x y z) = x + 2y + 3z\) at the point \((0, 2, -1, 0)\).
The normal direction is given by the gradient of \(F(x, y, z) = x + 2y + 3z - \sin(\pi x y z). \) We get 
\[ \nabla F(x, y, z) = \langle 1 - yz \cos(\pi x y z), 2 - xz \cos(\pi x y z), 3 - x y \cos(\pi x y z) \rangle, \text{ so } \nabla F(2, -1, 0) = \langle 1, 2, 5 \rangle. \] The normal line is given by 
\[ \mathbf{r}(t) = \langle 2 + t, -1 + 2t, 5t \rangle, \] and the tangent plane is given by 
\( (x - 2) + 2(y + 1) + 5z = 0. \)

8. Find the critical points and classify them for \(f(x, y) = (x^2 + y)e^{y/2}. \)
\[ \nabla f(x, y) = \langle 2xe^{y/2}, e^{y/2} + \frac{1}{2}(x^2 + y)e^{y/2} \rangle = e^{y/2}(2x, 1 + \frac{1}{2}(x^2 + y)). \] Setting this equal to zero we get 
\( x = 0 \) and \( 1 + \frac{1}{2}y = 0, \) so \( y = -2. \) Now 
\[ f_{xx} = 2e^{y/2}, \quad f_{xy} = f_{yx} = xe^{y/2}, \quad \text{and } f_{yy} = \frac{1}{2}e^{y/2}(1 + \frac{1}{2}(x^2 + y)) + \frac{1}{2}e^{y/2}, \] so at \( x = 0 \) and \( y = -2 \) we get 
\( f_{xx} = 2 > 0, \)
\[ f_{xy} = f_{yx} = 0, \text{ and } f_{yy} = \frac{1}{2}, \text{ and } D = f_{xx}f_{yy} - (f_{xy})^2 = 1 > 0, \text{ so we have a local minimum.} \]

9. Find the absolute maximum and minimum of \( f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2) \) on the disk \( x^2 + y^2 \leq 4. \)

By the Extreme Value Theorem the absolute maximum and minimum exist. In order to find all possible candidates inside the disk we need all zeros of the gradient \( \nabla f(x, y) = e^{-x^2-y^2}(-2x(x^2 + 2y^2) + 2x, -2y(x^2 + 2y^2) + 4y) = e^{-x^2-y^2}(2x(-x^2 - 2y^2 + 1), 2y(-x^2 - 2y^2 + 2)). \) The first component is zero if \( x = 0 \) or \( x^2 + 2y^2 = 1. \) The second component is zero if \( y = 0 \) or \( x^2 + 2y^2 = 2. \) From this we get the five solutions \((0, 0), (0, \pm 1), \) and \((\pm 1, 0). \) Plugging all of these into \( f \) we get \( f(0, 0) = 0, \ f(0, \pm 1) = 2e^{-1}, \) and \( f(\pm 1, 0) = e^{-1}. \)

In order to find possible candidates on the boundary it is probably easiest to parameterize it as \( x = 2 \cos t \) and \( y = 2 \sin t. \) Then \( f(2 \cos t, 2 \sin t) = e^{-4}(4 + 4 \sin^2 t), \) which has minimum and maximum values at \( t = 0 \) and \( t = \pi/2, \) respectively. (The same values repeat at \( t = \pi \) and \( t = 3\pi/2, \) but this is irrelevant for the question.) The values are \( f(2, 0) = 4e^{-4} \) and \( f(0, 2) = 8e^{-4}, \) respectively.

Comparing all possible candidates, the maximum is the largest value \( f(0, \pm 1) = 2e^{-1} = .74, \) and the minimum is the smallest value \( f(0, 0) = 0. \)

10. Find \( \int_D \frac{1}{1+x^2} \, dA \) where \( D \) is the triangular region with vertices \((0, 0), \ (1, 1), \) and \((0, 1). \)

We get \( \int_0^1 \int_0^1 (1 + x^2) \, dx \, dy = \int_0^1 (y + \frac{1}{2} y^3) \, dy = \frac{1}{2} + \frac{1}{12} = \frac{7}{12} = .58. \)

11. Find \( \iiint_H z^3 \sqrt{x^2 + y^2 + z^2} \, dV \) where \( H \) is the solid hemisphere that lies above the \( xy \)-plane and has center the origin and radius 1.

Using spherical coordinates, we get
\[
\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (r \cos \phi)^3 \cdot r \cdot r^2 \sin \phi \, d\phi \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \int_0^{\pi/2} r^6 \cos^3 \phi \sin \phi \, d\phi \, dr \, d\theta
\]
\[
= \int_0^{2\pi} \int_0^1 r^6 \left[ -\frac{1}{4} \cos^4 \phi \right]_{\phi=0}^{\phi=\pi/2} \, dr \, d\theta
\]
\[
= \int_0^{2\pi} \int_0^1 \frac{1}{4} r^6 \, dr \, d\theta
\]
\[
= \frac{2\pi}{4 \cdot \frac{7}{4}} = \frac{\pi}{2} = .22.
\]

12. Evaluate the line integral \( \int_C x \, ds \) where \( C \) is the arc of the parabola \( y = x^2 \) from \((0, 0) \) to \((1, 1). \)

Using the parameterization \( x(t) = t, y(t) = t^2, 0 \leq t \leq 1, \) we get
\[
\int_0^1 x(t) \sqrt{x'(t)^2 + y'(t)^2} \, dt = \int_0^1 t \sqrt{1 + 4t^2} \, dt
\]
\[
= \left[ \frac{1}{12} (1 + 4t^2)^{3/2} \right]_0^1 = \frac{5^{3/2} - 1}{12} = .85.
\]

13. Show that \( F(x, y) = (4x^3 + 3xy^3, 3x^4 - y^3) \) is conservative, and use this fact to evaluate \( \int_C F \cdot dr \) along the curve \( r(t) = (t + \sin \pi t, 2t + \cos \pi t), 0 \leq t \leq 1. \)
Denoting the components of $\mathbf{F}$ by $P$ and $Q$, one easily checks $P_y = Q_x$, so $\mathbf{F}$ is conservative. Integration leads to the potential $f(x, y) = x^4 y^2 - x^2 y^3 + y^4$. The curve has initial point $\mathbf{r}(0) = (0, 1)$ and endpoint $\mathbf{r}(1) = (1, 1)$, so the value of the integral is $f(1, 1) - f(0, 1) = 1 - 1 = 0$.

14. Use Green’s Theorem to evaluate $\int_C \sqrt{1 + x^3} \, dx + 2xy \, dy$, where $C$ is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$. (I.e., it is the boundary of the triangle, parameterized in positive orientation.)

If $D$ denotes the inside of the triangle, we get $\int\int_D (2y - 0) \, dA = \int_0^1 \int_0^{3x} 2y \, dy \, dx = \int_0^1 9x^2 \, dx = 3$. 