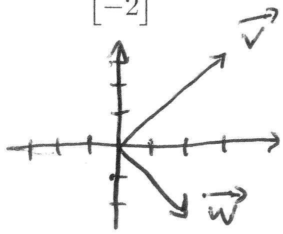
HOMEWORK KEY M221
CHAPTER 1

SECTION 1.1

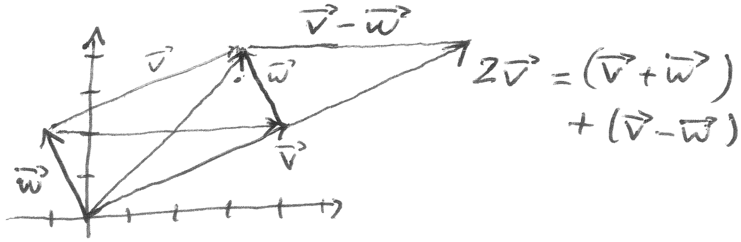
3. If $\mathbf{v}+\mathbf{w} = \begin{bmatrix}5\\1\end{bmatrix}$ and $\mathbf{v}-\mathbf{w} = \begin{bmatrix}1\\5\end{bmatrix}$, compute and draw $\mathbf{v}$ and $\mathbf{w}$.

Solution: $\mathbf{v} = \begin{bmatrix}3\\3\end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix}2\\-2\end{bmatrix}$.

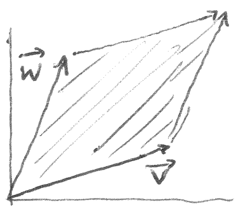


8. The parallelogram in Figure 1.1 has diagonal $\mathbf{v}+\mathbf{w}$. What is its other diagonal? What is the sum of the two diagonals? Draw that vector sum.

Solution: The other diagonal is $\mathbf{v}-\mathbf{w}$ (or $\mathbf{w}-\mathbf{v}$). The sum of the two diagonals is $2\mathbf{v}$ (or $2\mathbf{w}$).



18. (referring to Figure 1.5 (a)) Restricted by $0 \le c \le 1$ and $0 \le d \le 1$, shade in all combinations $c\mathbf{v}+d\mathbf{w}$.



31. Write down three equations for c, d, e so that $c\mathbf{u}+d\mathbf{v}+e\mathbf{w} = \mathbf{b}$. Can you somehow find c, d, and e?

$$\mathbf{u} = \begin{bmatrix}2\\-1\\0\end{bmatrix} \quad \mathbf{v} = \begin{bmatrix}-1\\2\\-1\end{bmatrix} \quad \mathbf{w} = \begin{bmatrix}0\\-1\\2\end{bmatrix} \quad \mathbf{b} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

Solution: The equations are

$$2c - d = 1, \quad -c + 2d - e = 0, \quad -d + 2e = 0.$$

The solution is $c = \frac{3}{4}$, $d = \frac{1}{2}$, $e = \frac{1}{4}$. (There are various different ways to find this.)

Section 1.2

1. Calculate the dot products $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\cdot\mathbf{w}$ and $\mathbf{u}\cdot\mathbf{v}+\mathbf{w}$ and $\mathbf{w}\cdot\mathbf{v}$.

$$\mathbf{u} = \begin{bmatrix} -.6 \\ .8 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}.$$

Solution: $\mathbf{u}\cdot\mathbf{v} = 1.4$, $\mathbf{u}\cdot\mathbf{w} = 0$, $\mathbf{u}\cdot\mathbf{v}+\mathbf{w} = 1.4$, $\mathbf{w}\cdot\mathbf{v} = 48$.

2. Compute the lengths $\|u\|$ and $\|v\|$ and $\|w\|$ of those vectors. Check the Schwarz inequalities $|\mathbf{u}\cdot\mathbf{v}| \le \|u\|\,\|v\|$ and $|\mathbf{v}\cdot\mathbf{w}| \le \|v\|\,\|w\|$.

Solution $\|u\| = 1$, $\|v\| = 5$, $\|w\| = 10$. Check $|1.4| \le 1\cdot 5$ and $|48| \le 5\cdot 10$. Both inequalities ($1.4 < 5$ and $48 < 50$) are true.

7. Find the angle θ (from its cosine) between these pairs of vectors:

(a) $\mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution: $\mathbf{v}\cdot\mathbf{w} = 1\cdot 1 + \sqrt{3}\cdot 0 = 1$, $\|v\| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$, $\|w\| = 1$, and $\theta = \cos^{-1}\frac{\mathbf{v}\cdot\mathbf{w}}{\|v\|\,\|w\|} = \cos^{-1}\frac{1}{2} = \frac{\pi}{3} = 60°$.

(b) $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

Solution: $\mathbf{v}\cdot\mathbf{w} = 2\cdot 2 + 2\cdot(-1) + (-1)\cdot 2 = 0$, so these vectors are perpendicular, i.e., $\theta = \frac{\pi}{2} = 90°$.

(c) $\mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$.

Solution: $\mathbf{v}\cdot\mathbf{w} = 1\cdot(-1) + \sqrt{3}\cdot\sqrt{3} = 2$, $\|v\| = \|w\| = 2$, and $\theta = \cos^{-1}\frac{\mathbf{v}\cdot\mathbf{w}}{\|v\|\,\|w\|} = \cos^{-1}\frac{1}{2} = \frac{\pi}{3} = 60°$.

(d) $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$.

Solution: $\mathbf{v}\cdot\mathbf{w} = 3\cdot(-1) + 1\cdot(-2) = -5$, $\|v\| = \sqrt{3^2+1^2} = \sqrt{10}$, $\|w\| = \sqrt{(-1)^2+(-2)^2} = \sqrt{5}$, and $\theta = \cos^{-1}\frac{\mathbf{v}\cdot\mathbf{w}}{\|v\|\,\|w\|} = \cos^{-1}\frac{-5}{\sqrt{5}\sqrt{10}} = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4} = 135°$.

13. Find two vectors that are perpendicular to $(1,0,1)$ and to each other.

Solution: There are lots of solutions, one example is $(1,0,-1)$ and $(0,1,0)$.

27. If $\|\mathbf{v}\| = 5$ and $\|\mathbf{w}\| = 3$, what are the smallest and largest values of $\|\mathbf{v}-\mathbf{w}\|$? What are the smallest and largest values of $\mathbf{v}\cdot\mathbf{w}$?

Solution: Largest and smallest values for $\|\mathbf{v}-\mathbf{w}\|$ are 8 and 2, respectively. For $\mathbf{v}\cdot\mathbf{w}$ they are 15 and -15, respectively. For both questions the extreme cases are when the vectors point in the same or in opposite directions, respectively. (Now I have definitely overused the word "respectively"...)

Section 1.3

6. Which values of c give dependent columns (combination equals zero)?

$$\begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & c \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & c \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} c & c & c \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$$

Solution: $c = 3$, $c = -1$, and $c = 0$, respectively.

10. A *forward* difference matrix Δ is upper triangular:

$$\Delta \mathbf{z} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_2 - z_1 \\ z_3 - z_2 \\ 0 - z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \mathbf{b}.$$

Find z_1, z_2, z_3 from b_1, b_2, b_3. What is the inverse matrix $\mathbf{z} = \Delta^{-1}\mathbf{b}$?

Solution: $z_3 = -b_3$, $z_2 = z_3 - b_2 = -b_3 - b_2$, and $z_1 = z_2 - b_1 = -b_3 - b_2 - b_1$. The inverse matrix is

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -b_1 - b_2 - b_3 \\ -b_2 - b_3 \\ -b_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \Delta^{-1}\mathbf{b}.$$