HOMEWORK KEY M 221

SECTIONS 2.1 - 2.5

## SECTION 2.1

**3.** If equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the vectors in the column picture, the coefficient matrix, the solution? The new equations in Problem 1 would be $x = 2$, $x + y = 5$, $z = 4$.

Solution: One of the planes in the row picture changes, two of the vectors in the column picture change, one entry in the coefficient matrix changes, the solution does not change.

**9.** Compute each $A\mathbf{x}$ by dot products of the rows with the column vector:

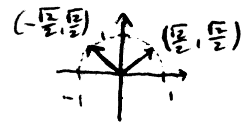
Solution:

$$\text{(a)} \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 18 \\ 5 \\ 0 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 5 \end{bmatrix}$$

**10.** Compute each $A\mathbf{x}$ in Problem 9 as a combination of the columns. How many separate multiplications for $A\mathbf{x}$, when the matrix is "3 by 3"?

Solution: The answer is obviously the same as in Problem 9. The number of multiplications is 3 for each entry in the resulting vector, so 9 multiplications total for all three entries.

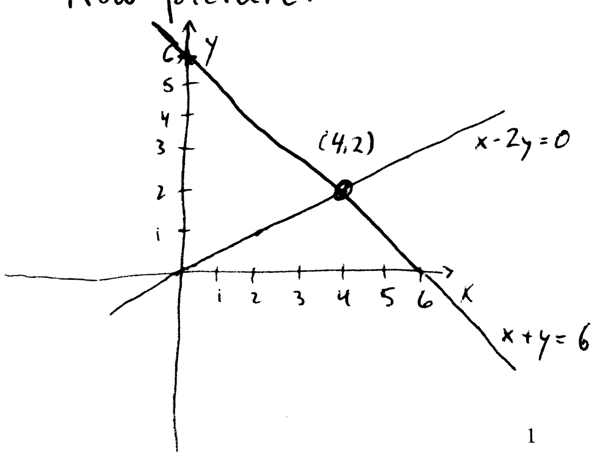
**21.** What 2 by 2 matrix $R$ rotates every vector through 45°? The vector $(1, 0)$ goes to $(\sqrt{2}/2, \sqrt{2}/2)$. The vector $(0, 1)$ goes to $(-\sqrt{2}/2, \sqrt{2}/2)$. Those determine the matrix. Draw these particular vectors in the $xy$ plane and find $R$.



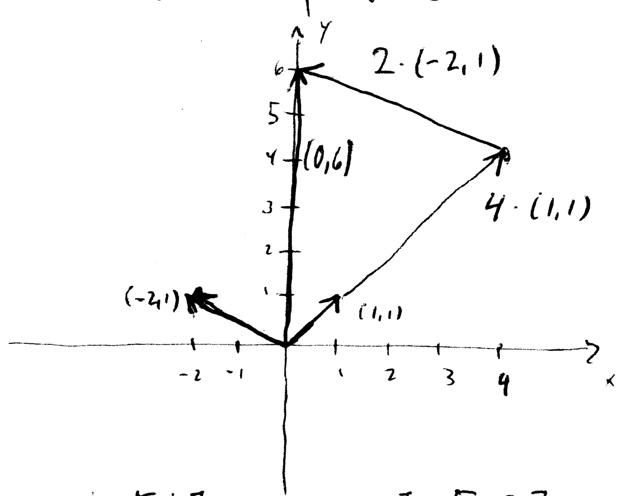

Solution: $R = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$.

**26.** Draw the row and column pictures for the equations $x - 2y = 0$, $x + y = 6$.

Row picture:




Column picture




$$4 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

## SECTION 2.2

**1.** What multiple $l_{21}$ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} 2x + 3y &= 1 \\ 10x + 9y &= 11 \end{aligned}$$

After this elimination step, write down the upper triangular system and circle the two pivots. The numbers 1 and 11 have no influence on the pivots.

Solution: $l_{21} = 5$. Upper triangular form with pivots in bold:

$$\begin{aligned} \mathbf{2}x + 3y &= 1 \\ -\mathbf{6}y &= 6 \end{aligned}$$

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**2.** Solve the triangular system of Problem 1 by back substitution, $y$ before $x$. Verify that $x$ times $(2, 10)$ plus $y$ times $(3, 9)$ equals $(1, 11)$. If the right side changes to $(4, 44)$, what is the new solution?

Solution: $y = -1$, $x = 2$. If the right side (in the original equations) changes to $(4, 44)$, the new solution is $y = -4$, $x = 8$.

**5.** Choose a right side which gives no solution and another right side which gives infinitely many solutions. What are two of those solutions?

$$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= \end{aligned}$$

Solution: The right side 20 gives infinitely many solutions, any other number gives no solution. Two of the solutions with right side 20 are $x = 0$, $y = 5$ and $x = 2$, $y = 2$ (but obviously there are lots of other solutions, since there are infinitely many.)

**11.** A system of linear equations can't have exactly two solutions. Why?

(a) If $(x, y, z)$ and $(X, Y, Z)$ are two solutions, what is another solution?

(b) If 25 planes meet at two points, where else do they meet?

Solution: (a) Any point on the line between the two points is another solution. E.g., the midpoint $((x+X)/2, (y+Y)/2, (z+Z)/2)$ is one such point. This also answers the first question why there can't be exactly two solutions.

(b) Again, they meet along the whole line between the two points.

**12.** Reduce this system to upper triangular form by two row operations.

$$\begin{array}{rcrcrcr} 2x & + & 3y & + & z & = & 8 \\ 4x & + & 7y & + & 5z & = & 20 \\ & & -2y & + & 2z & = & 0 \end{array}$$

Circle the pivots. Solve by back substitution for $z, y, x$.

Solution: First subtract twice the first equation from the second equation:

$$\begin{array}{rcrcrcl} 2x & + & 3y & + & z & = & 8 \\ & & y & + & 3z & = & 4 \\ & & -2y & + & 2z & = & 0 \end{array}$$

Now add twice the second equation to the third equation to give the upper triangular form, pivots in bold:

$$\begin{array}{rcrcrcl} \mathbf{2}x & + & 3y & + & z & = & 8 \\ & & \mathbf{1}y & + & 3z & = & 4 \\ & & & & \mathbf{8}z & = & 8 \end{array}$$

Back substitution gives $z = 1$, $y = 1$, $x = 2$.

**21.** Find the pivots and the solution for both systems ($A\mathbf{x} = \mathbf{b}$ and $K\mathbf{x} = \mathbf{b}$):

Solution: We start with the first system.

$$\begin{array}{rcrcrcrcl} 2x & + & y & & & & & = & 0 \\ x & + & 2y & + & z & & & = & 0 \\ & & y & + & 2z & + & t & = & 0 \\ & & & & z & + & 2t & = & 5 \end{array}$$

First subtract $1/2$ of the first row from the second row:

$$\begin{array}{rcrcrcrcl} 2x & + & y & & & & & = & 0 \\ & & \frac{3}{2}y & + & z & & & = & 0 \\ & & y & + & 2z & + & t & = & 0 \\ & & & & z & + & 2t & = & 5 \end{array}$$

Now subtract $2/3$ of the second row from the third row:

$$\begin{array}{rcrcrcrcl} 2x & + & y & & & & & = & 0 \\ & & \frac{3}{2}y & + & z & & & = & 0 \\ & & & & \frac{4}{3}z & + & t & = & 0 \\ & & & & z & + & 2t & = & 5 \end{array}$$

Lastly, subtract $3/4$ of the third row from the fourth row, pivots in bold:

$$\begin{array}{rcrcrcrcl} \mathbf{2}x & + & y & & & & & = & 0 \\ & & \mathbf{\frac{3}{2}}y & + & z & & & = & 0 \\ & & & & \mathbf{\frac{4}{3}}z & + & t & = & 0 \\ & & & & & & \mathbf{\frac{5}{4}}t & = & 5 \end{array}$$

Back substitution gives $t = 4$, $z = -3$, $y = 2$, $x = -1$.

Now the second system:

$$\begin{array}{rcrcrcrcl} 2x & - & y & & & & & = & 0 \\ -x & + & 2y & - & z & & & = & 0 \\ & & -y & + & 2z & - & t & = & 0 \\ & & & & -z & + & 2t & = & 5 \end{array}$$

First add $1/2$ of the first row to the second row:

$$\begin{array}{rcrcrcrcl} 2x & - & y & & & & & = & 0 \\ & & \frac{3}{2}y & - & z & & & = & 0 \\ & & -y & + & 2z & - & t & = & 0 \\ & & & & -z & + & 2t & = & 5 \end{array}$$

Now add $2/3$ of the second row to the third row:

$$\begin{array}{rcrcrcrcl} 2x & - & y & & & & & = & 0 \\ & & \frac{3}{2}y & - & z & & & = & 0 \\ & & & & \frac{4}{3}z & - & t & = & 0 \\ & & & & -z & + & 2t & = & 5 \end{array}$$

Lastly, add $3/4$ of the third row to the fourth row, pivots in bold:

$$\begin{array}{rcrcrcrcl} \mathbf{2}x & - & y & & & & & = & 0 \\ & & \mathbf{\frac{3}{2}}y & - & z & & & = & 0 \\ & & & & \mathbf{\frac{4}{3}}z & - & t & = & 0 \\ & & & & & & \mathbf{\frac{5}{4}}t & = & 5 \end{array}$$

Back substitution gives $t = 4$, $z = 3$, $y = 2$, $x = 1$.

## SECTION 2.3

**4.** Include $\mathbf{b} = (1, 0, 0)$ as a fourth column in Problem 3 to produce $[A\ \mathbf{b}]$. Carry out the elimination steps on this augmented matrix to solve $A\mathbf{x} = \mathbf{b}$.

Solution:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 4 & 6 & 1 & 0 \\ -2 & 2 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -4 \\ 0 & 4 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & -2 & 10 \end{bmatrix}$$

Back substitution gives $z = -5$, $y = 1/2$, $x = 1/2$.

## SECTION 2.4

**1.** $A$ is 3 by 5, $B$ is 5 by 3, $C$ is 5 by 1, and $D$ is 3 by 1. All entries are 1. Which of these matrix operations is allowed and what are the results?

Solution: $BA$ is a 5 by 5 matrix where all entries are 3. $AB$ is a 3 by 3 matrix where all entries are 5. $ABD$ is a 5 by 1 matrix (i.e., a column vector) where all entries are 15. $DBA$ and $A(B+C)$ are not defined, the dimensions of the matrices do not match.

**6.** Show that $(A+B)^2$ is different from $A^2+2AB+B^2$ when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

Write down the correct rule for $(A+B)^2 = A^2 + \_\_\_\_ + B^2$.

Solution:

$$(A+B)^2 = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix}$$

and

$$A^2+2AB+B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$$

The correct rule is $(A+B)^2 = A^2 + AB + BA + B^2$.

**14.** True or false?

(a) If $A^2$ is defined, then $A$ is necessarily square.

(b) If $AB$ and $BA$ are defined, then $A$ and $B$ are square.

(c) If $AB$ and $BA$ are defined, then $AB$ and $BA$ are square.

(d) If $AB = B$, then $A = I$.

Solution: (a) True, in order to multiply $A$ with itself, the number of columns and rows has to be equal.

(b) False, these products are always defined when $A$ is $m$ by $n$ and $B$ is $n$ by $m$. E.g., $A$ could be 2 by 3, and $B$ could be 3 by 2.

(c) True. If these products are defined, then $A$ is $m$ by $n$ and $B$ is $n$ by $m$, so $AB$ is $m$ by $m$ and $BA$ is $n$ by $n$.

(d) False, this is also true if $B = 0$ (and there are many more counterexamples).

## Section 2.5

**2.** For these "permutation matrices" find $P^{-1}$ by trial and error (with 1's and 0's):

Solution:

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ is its own inverse, } P^{-1} = P.$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ has inverse } P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

**9.** Suppose $A$ is invertible and you exchange its first two rows to reach $B$. Is the new matrix invertible and how would you find $B^{-1}$ from $A^{-1}$?

Solution: Yes, $B$ is invertible, and you get $B^{-1}$ from $A^{-1}$ by exchanging its first two columns.

**10.** Find the inverses of

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}.$$

Solution: For the first matrix it is easy to guess that $A^{-1}$ has entries on the same diagonal as $A$, and that the entries are reciprocals of the entries of $A$. Trial and error (or various sophisticated arguments) show that the order of the elements is reversed, i.e.,

$$A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}.$$

The second matrix is a block matrix of the form $\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ with 2 by 2 blocks, so its inverse is $\begin{bmatrix} C^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}$. We know the formula for inverting 2 by 2 matrices, so

$$B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}.$$

**29.** True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.

(b) Every matrix with 1's down the main diagonal is invertible.

(c) If $A$ is invertible, then $A^{-1}$ and $A^2$ are invertible.

Solution: (a) True. If $A$ has zeros in the $i$-th row, then the $i$-th row of $AB$ is always zero (because it is a linear combination of the rows of $B$ with coefficients in the $i$-th row of A, which in this case are all zero). In particular, $AB$ can never be the identity matrix, no matter what $B$ is.

(b) False. Counterexample $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible.

(c) True. The inverse of $A^{-1}$ is $A$, and the inverse of $A^2$ is $(A^{-1})^2$.

**40.** $A$ is a 4 by 4 matrix with 1's on the diagonal and $-a$, $-b$, $-c$ on the diagonal above. Find $A^{-1}$ for this bidiagonal matrix.

Solution:

$$\begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$