### Section 2.1

3. If equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the vectors in the column picture, the coefficient matrix, the solution? The new equations in Problem 1 would be \( x = 2, \ x + y = 5, \ z = 4 \).

Solution: One of the planes in the row picture changes, two of the vectors in the column picture change, one entry in the coefficient matrix changes, the solution does not change.

9. Compute each \( A \mathbf{x} \) by dot products of the rows with the column vector:

Solution:

\[
\begin{align*}
(a) \quad & \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 18 \\ 5 \end{bmatrix} = 18 \\
(b) \quad & \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 5 \\
\end{align*}
\]

10. Compute each \( A \mathbf{x} \) in Problem 9 as a combination of the columns. How many separate multiplications for \( A \mathbf{x} \), when the matrix is "3 by 3"?

Solution: The answer is obviously the same as in Problem 9. The number of multiplications is 3 for each entry in the resulting vector, so 9 multiplications total for all three entries.

21. What 2 by 2 matrix \( R \) rotates every vector through 45°? The vector \((1, 0)\) goes to \((\sqrt{2}/2, \sqrt{2}/2)\). The vector \((0, 1)\) goes to \((-\sqrt{2}/2, \sqrt{2}/2)\). Those determine the matrix. Draw these particular vectors in the \( xy \) plane and find \( R \).

Solution: \( R = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \).

26. Draw the row and column pictures for the equations \( x - 2y = 0, \ x + y = 6 \).

Row picture:

Column picture:

\[ 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} \]
SECTION 2.2

1. What multiple $l_{21}$ of equation 1 should be subtracted from equation 2?

\[
2x + 3y = 1 \\
10x + 9y = 11
\]

After this elimination step, write down the upper triangular system and circle the two pivots. The numbers 1 and 11 have no influence on the pivots.
Solution: $l_{21} = 5$. Upper triangular form with pivots in bold:

\[
2x + 3y = 1 \\
\quad -6y = 6
\]

2. Solve the triangular system of Problem 1 by back substitution, $y$ before $x$. Verify that $x$ times $(2,10)$ plus $y$ times $(3,9)$ equals $(1,11)$. If the right side changes to $(4,44)$, what is the new solution?
Solution: $y = -1$, $x = 2$. If the right side (in the original equations) changes to $(4,44)$, the new solution is $y = -4$, $x = 8$.

5. Choose a right side which gives no solution and another right side which gives infinitely many solutions. What are two of those solutions?

\[
3x + 2y = 10 \\
6x + 4y =
\]

Solution: The right side 20 gives infinitely many solutions, any other number gives no solution. Two of the solutions with right side 20 are $x = 0$, $y = 5$ and $x = 2$, $y = 2$ (but obviously there are lots of other solutions, since there are infinitely many.)

11. A system of linear equations can’t have exactly two solutions. Why?
(a) If $(x,y,z)$ and $(X,Y,Z)$ are two solutions, what is another solution?
(b) If 25 planes meet at two points, where else do they meet?
Solution: (a) Any point on the line between the two points is another solution. E.g., the midpoint $((x+X)/2, (y+Y)/2, (z+Z)/2)$ is one such point. This also answers the first question why there can’t be exactly two solutions.
(b) Again, they meet along the whole line between the two points.

12. Reduce this system to upper triangular form by two row operations.

\[
2x + 3y + z = 8 \\
4x + 7y + 5z = 20 \\
-2y + 2z = 0
\]

Circle the pivots. Solve by back substitution for $z, y, x$. 

Solution: First subtract twice the first equation from the second equation:

\[
\begin{align*}
2x + 3y + z &= 8 \\
y + 3z &= 4 \\
-2y + 2z &= 0
\end{align*}
\]

Now add twice the second equation to the third equation to give the upper triangular form, pivots in bold:

\[
\begin{align*}
2x + 3y + z &= 8 \\
1y + 3z &= 4 \\
8z &= 8
\end{align*}
\]

Back substitution gives \( z = 1, y = 1, x = 2 \).

21. Find the pivots and the solution for both systems (\( Ax = b \) and \( Kx = b \)):

Solution: We start with the first system.

\[
\begin{align*}
2x + y &= 0 \\
x + 2y + z &= 0 \\
y + 2z + t &= 0 \\
z + 2t &= 5
\end{align*}
\]

First subtract 1/2 of the first row from the second row:

\[
\begin{align*}
2x + y &= 0 \\
\frac{3}{2}y + z &= 0 \\
y + 2z + t &= 0 \\
z + 2t &= 5
\end{align*}
\]

Now subtract 2/3 of the second row from the third row:

\[
\begin{align*}
2x + y &= 0 \\
\frac{3}{2}y + z &= 0 \\
\frac{4}{3}z + t &= 0 \\
z + 2t &= 5
\end{align*}
\]

Lastly, subtract 3/4 of the third row from the fourth row, pivots in bold:

\[
\begin{align*}
2x + y &= 0 \\
\frac{3}{2}y + z &= 0 \\
\frac{4}{3}z + t &= 0 \\
\frac{5}{4}t &= 5
\end{align*}
\]

Back substitution gives \( t = 4, z = -3, y = 2, x = -1 \).
Now the second system:
\[
\begin{align*}
2x - y &= 0 \\
-x + 2y - z &= 0 \\
-y + 2z - t &= 0 \\
-z + 2t &= 5
\end{align*}
\]
First add 1/2 of the first row to the second row:
\[
\begin{align*}
2x - y &= 0 \\
\frac{3}{2}y - z &= 0 \\
-y + 2z - t &= 0 \\
-z + 2t &= 5
\end{align*}
\]
Now add 2/3 of the second row to the third row:
\[
\begin{align*}
2x - y &= 0 \\
\frac{3}{2}y - z &= 0 \\
\frac{4}{3}z - t &= 0 \\
-z + 2t &= 5
\end{align*}
\]
Lastly, add 3/4 of the third row to the fourth row, pivots in bold:
\[
\begin{align*}
2x - y &= 0 \\
\frac{3}{2}y - z &= 0 \\
\frac{4}{3}z - t &= 0 \\
\frac{5}{4}t &= 5
\end{align*}
\]
Back substitution gives \( t = 4, z = 3, y = 2, x = 1 \).

**Section 2.3**

4. Include \( b = (1, 0, 0) \) as a fourth column in Problem 3 to produce \([A \ b]\). Carry out the elimination steps on this augmented matrix to solve \( Ax = b \).

Solution:
\[
\begin{bmatrix}
1 & 1 & 0 & 1 \\
4 & 6 & 1 & 0 \\
-2 & 2 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 2 & 1 & -4 \\
0 & 4 & 0 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 2 & 1 & -4 \\
0 & 0 & -2 & 10
\end{bmatrix}
\]
Back substitution gives \( z = -5, y = 1/2, x = 1/2 \).

**Section 2.4**

1. \( A \) is 3 by 5, \( B \) is 5 by 3, \( C \) is 5 by 1, and \( D \) is 3 by 1. All entries are 1. Which of these matrix operations is allowed and what are the results?

Solution: \( BA \) is a 5 by 5 matrix where all entries are 3. \( AB \) is a 3 by 3 matrix where all entries are 5. \( ABD \) is a 5 by 1 matrix (i.e., a column
vector) where all entries are 15. DBA and A(B + C) are not defined, the dimensions of the matrices do not match. 

6. Show that \((A + B)^2\) is different from \(A^2 + 2AB + B^2\) when 

\[
A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.
\]

Write down the correct rule for \((A + B)^2 = A^2 + \_\_\_ + B^2\).

Solution:

\[
(A + B)^2 = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix}
\]

and 

\[
A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + 2 \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}
\]

The correct rule is \((A + B)^2 = A^2 + AB + BA + B^2\).

14. True or false?

(a) If \(A^2\) is defined, then \(A\) is necessarily square.

(b) If \(AB\) and \(BA\) are defined, then \(A\) and \(B\) are square.

(c) If \(AB\) and \(BA\) are defined, then \(AB\) and \(BA\) are square.

(d) If \(AB = B\), then \(A = I\).

Solution: (a) True, in order to multiply \(A\) with itself, the number of columns and rows has to be equal.

(b) False, these products are always defined when \(A\) is \(m\) by \(n\) and \(B\) is \(n\) by \(m\). E.g., \(A\) could be 2 by 3, and \(B\) could be 3 by 2.

(c) True. If these products are defined, then \(A\) is \(m\) by \(n\) and \(B\) is \(n\) by \(m\), so \(AB\) is \(m\) by \(m\) and \(BA\) is \(n\) by \(n\).

(d) False, this is also true if \(B = 0\) (and there are many more counterexamples).

**Section 2.5**

2. For these “permutation matrices” find \(P^{-1}\) by trial and error (with 1's and 0's):

Solution:

\[
P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
\]

is its own inverse, \(P^{-1} = P\).

\[
P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}
\]

has inverse \(P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\).
9. Suppose $A$ is invertible and you exchange its first two rows to reach $B$. Is the new matrix invertible and how would you find $B^{-1}$ from $A^{-1}$?

Solution: Yes, $B$ is invertible, and you get $B^{-1}$ from $A^{-1}$ by exchanging its first two columns.

10. Find the inverses of

\[
A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}.\]

Solution: For the first matrix it is easy to guess that $A^{-1}$ has entries on the same diagonal as $A$, and that the entries are reciprocals of the entries of $A$. Trial and error (or various sophisticated arguments) show that the order of the elements is reversed, i.e.,

\[
A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}.\]

The second matrix is a block matrix of the form $\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ with 2 by 2 blocks, so its inverse is $\begin{bmatrix} C^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}$. We know the formula for inverting 2 by 2 matrices, so

\[
B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}.\]

29. True or false (with a counterexample if false and a reason if true):

(a) A 4 by 4 matrix with a row of zeros is not invertible.

(b) Every matrix with 1's down the main diagonal is invertible.

(c) If $A$ is invertible, then $A^{-1}$ and $A^2$ are invertible.

Solution: (a) True. If $A$ has zeros in the $i$-th row, then the $i$-th row of $AB$ is always zero (because it is a linear combination of the rows of $B$ with coefficients in the $i$-th row of $A$, which in this case are all zero). In particular, $AB$ can never be the identity matrix, no matter what $B$ is.

(b) False. Counterexample $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible.

(c) True. The inverse of $A^{-1}$ is $A$, and the inverse of $A^2$ is $(A^{-1})^2$. 
40. $A$ is a 4 by 4 matrix with 1’s on the diagonal and $-a$, $-b$, $-c$ on the diagonal above. Find $A^{-1}$ for this bidiagonal matrix.

Solution:

$$
\begin{bmatrix}
1 & -a & 0 & 0 \\
0 & 1 & -b & 0 \\
0 & 0 & 1 & -c \\
0 & 0 & 0 & 1
\end{bmatrix}^{-1} =
\begin{bmatrix}
1 & a & ab & abc \\
0 & 1 & b & bc \\
0 & 0 & 1 & c \\
0 & 0 & 0 & 1
\end{bmatrix}
$$