Section 3.1

2. Suppose the multiplication $c \mathbf{x}$ is defined to produce $(cx_1, 0)$ instead of $(cx_1, cx_2)$, which of the eight conditions are not satisfied?

Solution: 3, 4, 5

10. Which of the following subsets of $\mathbb{R}^3$ are actually subspaces?

(a) The plane of vectors $(b_1, b_2, b_3)$ with $b_1 = b_2$.
(b) The plane of vectors with $b_1 = 1$.
(c) The vectors with $b_1 b_2 b_3 = 0$.
(d) All linear combinations of $\mathbf{v} = (1, 4, 0)$ and $\mathbf{w} = (2, 2, 2)$.
(e) All vectors that satisfy $b_1 + b_2 + b_3 = 0$.
(f) All vectors with $b_1 \leq b_2 \leq b_3$.

Solution: a, d, e

19. Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. $$

Solution: $C(A)$ is a line (the $x$-axis), $C(B)$ is a plane (the $xy$-plane), $C(C)$ is a line (through 0 and $(1, 2, 0)$).

Section 3.2

1. Reduce these matrices to their ordinary echelon forms $U$:

(a) $A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$

(b) $B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$.

Solution:

(a) $U_A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(b) $U_B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$.

3. By combining the special solutions in Problem 2, describe every solution to $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$. The nullspace contains only $\mathbf{x} = \mathbf{0}$ when there are no _____.

Solution: All solutions to $A\mathbf{x} = \mathbf{0}$ are given by $\mathbf{x} = x_2(-2, 1, 0, 0, 0) + x_4(0, 0, -2, 1, 0) + x_5(0, 0, -3, 0, 1)$, all solutions to $B\mathbf{x} = \mathbf{0}$ are given by $\mathbf{x} = x_3(1, -1, 1)$. The nullspace contains only $\mathbf{x} = \mathbf{0}$ when there are no free variables.

9. True or false (with reason if true or example to show it is false):

(a) A square matrix has no free variables.

(b) An invertible matrix has no free variables.

(c) An $m$ by $n$ matrix has no more than $n$ pivot variables.

(d) An $m$ by $n$ matrix has no more than $m$ pivot variables.
Solution: (a) False, e.g., \( A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \) has two free variables. (b) True, because an invertible matrix has only 0 in its nullspace. (c) True because there are only \( n \) variables total. (d) True because every row contains at most one pivot.

31. If the nullspace of \( A \) consists of all multiples of \( x = (2, 1, 0, 1) \), how many pivots appear in \( U \)? What is \( R \)?

Solution: 3 pivots appear in \( U \), and \( R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \) in block form, where \( I \) is the 3 by 3 identity matrix, and \( F = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} \). The 0 blocks have 3 and 1 columns, respectively, and they could have any number of rows.

Section 3.3

2. Find the reduced row echelon form \( R \) and the rank of these matrices:

(a) The 3 by 4 matrix with all entries equal to 4.

(b) The 3 by 4 matrix with \( a_{ij} = i + j - 1 \).

(c) The 3 by 4 matrix with \( a_{ij} = (-1)^j \).

Solution: (a) Rank 1,

\[
R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

(b) Rank 2,

\[
R = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

(c) Rank 1,

\[
R = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

8. Fill out these matrices so that they have rank 1:

\[
A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 \\ 4 & 8 & 16 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 9 \\ 1 & 2 & 6 & -3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} a & b \\ c \end{bmatrix}.
\]

Solution:

\[
A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 9 & -9/2 \\ 1 & 3 & -3/2 \\ 2 & 6 & -3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} a & b \\ c & bc/a \end{bmatrix}.
\]

12. If \( A \) has rank \( r \), then it has an \( r \) by \( r \) submatrix \( S \) that is invertible. Remove \( m - r \) rows and \( n - r \) columns to find an invertible submatrix \( S \) inside \( A \), \( B \), and \( C \). You could keep the
pivot rows and pivot columns:

\[
A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Solution:

\[
S_A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \quad S_B = [1] \quad S_C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

24. What is the nullspace matrix \( N \) (containing the special solutions) for \( A, B, C \)?

\[
A = [I \ I] \quad \text{and} \quad B = [I \ I] \quad \text{and} \quad C = [I \ I \ I].
\]

Solution:

\[
N_A = [-I \ I] \quad \text{and} \quad N_B = [-I \ I] \quad \text{and} \quad N_C = [-I \ -I \ I].
\]

SECTION 3.4

1. Execute the six steps of Worked Example 3.4A to describe the column space and nullspace of \( A \) and the complete solution to \( A\mathbf{x} = \mathbf{b} \):

\[
A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}.
\]

Solution: First we reduce the augmented matrix to row echelon form (step 1):

\[
\begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & -1 & -1 & -2 & b_3 - b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{bmatrix}.
\]

This shows that the condition for the existence of a solution is \( b_3 + b_2 - 2b_1 = 0 \) (step 2). The column space of \( A \) is the plane containing all linear combinations of the pivot columns \((2,2,2)\) and \((4,5,3)\), or equivalently all vectors \( \mathbf{b} \) satisfying the condition \( b_3 + b_2 - 2b_1 = 0 \) (step 3). The free variables are \( x_3 \) and \( x_4 \), so the first special solution is \( x_3 = 1, x_4 = 0 \), and by back substitution with \( U \) we get \( x_2 = -x_3 - 2x_4 = -1 \) and \( x_1 = \frac{1}{2} (-4x_2 - 6x_3 - 4x_4) = -1 \), so \( s_1 = (-1, -1, 1, 0) \). Similarly, the second special solution has \( x_3 = 0, x_4 = 1 \), and back substitution gives \( x_2 = -2 \) and \( x_1 = 2 \), so \( s_2 = (2, -2, 0, 1) \). The nullspace consists of all linear combinations \( \mathbf{x}_N = c_1 s_1 + c_2 s_2 \) (step 4). The particular solution to \( \mathbf{b} = (4,3,5) \) is obtained by setting \( x_3 = x_4 = 0 \) and solving by back substitution with \( U \), so \( x_2 = b_2 - b_1 - x_3 - x_4 = -1 \) and \( x_1 = \frac{1}{2} (b_1 - 4x_2 - 6x_3 - 4x_4) = 4 \), so \( \mathbf{x}_P = (4, -1, 0, 0) \). The complete solution is \( \mathbf{x} = \mathbf{x}_P + \mathbf{x}_N \) (step 5). Continuing the elimination to the reduced form gives

\[
\begin{bmatrix} 2 & 0 & 2 & -4 & 5b_1 - 4b_2 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & -2 & \frac{5}{2}b_1 - 2b_2 \\ 0 & 1 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 2b_1 \end{bmatrix}.
\]

This shows directly from the matrix that the particular solution is always \( (\frac{5}{2}b_1 - 2b_2, b_2 - b_1, 0, 0) \), and that the special solutions are \((-1, -1, 1, 0)\) and \((2, -2, 0, 1)\) (step 6).
5. Under what conditions on \( b_1, b_2, b_3 \) is this system solvable? Include \( \mathbf{b} \) as a fourth column in elimination. Find all solutions when that condition holds:

\[
\begin{align*}
    x + 2y - 2z &= b_1 \\
    2x + 5y - 4z &= b_2 \\
    4x + 9y - 8z &= b_3
\end{align*}
\]

Solution: Again we reduce the augmented matrix by elimination:

\[
\begin{bmatrix}
1 & 2 & -2 & b_1 \\
2 & 5 & -4 & b_2 \\
4 & 9 & -8 & b_3
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & -2 & b_1 \\
0 & 1 & 0 & b_2 - 2b_1 \\
0 & 1 & 0 & b_3 - 4b_1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & -2 & b_1 \\
0 & 1 & 0 & b_2 - 2b_1 \\
0 & 0 & 0 & b_3 - b_2 - 2b_1
\end{bmatrix}.
\]

The condition is \( b_3 - b_2 - 2b_1 = 0 \). Nullspace solutions are multiples of the special solution \( \mathbf{s} = (2, 0, 1) \), particular solution is \((5b_1 - 2b_2, b_2 - 2b_1, 0)\), so the complete solution when the condition holds is \((5b_1 - 2b_2, b_2 - 2b_1, 0) + x_3(2, 0, 1)\).

17. Solution: The largest possible rank of a 6 by 4 matrix is 4. Then there is a pivot in every column of \( \mathbf{U} \) and \( \mathbf{R} \). The solution to \( \mathbf{A}\mathbf{x} = \mathbf{b} \) is unique. The nullspace of \( \mathbf{A} \) is zero. An example is \( \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) (where \( I \) is the 4 by 4 identity matrix, and 0 is a 2 by 4 matrix of zeroes).

24. Give examples of matrices \( \mathbf{A} \) for which the number of solutions to \( \mathbf{A}\mathbf{x} = \mathbf{b} \) is

(a) 0 or 1, depending on \( \mathbf{b} \)
(b) \( \infty \), regardless of \( \mathbf{b} \)
(c) 0 or \( \infty \), depending on \( \mathbf{b} \)
(d) 1, regardless of \( \mathbf{b} \)

Solution: (a) \( \mathbf{A} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), (b) \( \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \), (c) \( \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \), (d) \( \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

31. Find matrices \( \mathbf{A} \) and \( \mathbf{B} \) with the given property or explain why you can't:

(a) The only solution of \( \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \) is \( \mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).
(b) The only solution of \( \mathbf{B}\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) is \( \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \).

Solution: (a) \( \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} \), (b) Impossible because \( \mathbf{B} \) would have to be a 2 by 3 matrix without free variables, i.e., of rank 3. However, the rank of a matrix can not be greater than the number of rows, in this example it could be at most 2.

34. Suppose you know that the 3 by 4 matrix \( \mathbf{A} \) has the vector \( \mathbf{s} = (2, 3, 1, 0) \) as the only special solution to \( \mathbf{A}\mathbf{x} = \mathbf{0} \).

(a) What is the rank of \( \mathbf{A} \) and the complete solution to \( \mathbf{A}\mathbf{x} = \mathbf{0} \)?
(b) What is the exact row reduced echelon form \( \mathbf{R} \) of \( \mathbf{A} \)?
(c) How do you know that \( \mathbf{A}\mathbf{x} = \mathbf{b} \) can be solved for all \( \mathbf{b} \)?

Solution: (a) The rank is 3, and the complete solution to \( \mathbf{A}\mathbf{x} = \mathbf{0} \) is \( \mathbf{x} = x_3\mathbf{s} = x_3(2, 3, 1, 0) \).
(b) \[ R = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

(c) The matrix has full row rank, so there always is at least one solution (and since there is a free variable, really infinitely many).

Section 3.5

1. Show that \( v_1 \), \( v_2 \), \( v_3 \) are independent but \( v_1 \), \( v_2 \), \( v_3 \), \( v_4 \) are dependent:

\[
\begin{align*}
v_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
v_4 &= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}
\end{align*}
\]

Solve \( c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0 \) or \( A x = 0 \). The \( v \)'s go in the columns of \( A \).

Solution: Any four vectors in \( \mathbb{R}^3 \) are dependent (since the dimension of \( \mathbb{R}^3 \) is 3), and the vectors \( v_1 \), \( v_2 \), \( v_3 \) are independent because they are columns of an invertible 3 by 3 matrix. Solving the system \( A x = 0 \) is finding the nullspace of the matrix

\[
A = \begin{bmatrix} 1 & 1 & 1 & 2 \\
0 & 1 & 1 & 3 \\
0 & 0 & 1 & 4 \end{bmatrix}
\]

Special solution here is \( x_4 = 1 \), \( x_3 = -4 \), \( x_2 = 1 \), \( x_1 = 1 \), so the nullspace consists of all \( (c_1, c_2, c_3, c_4) = x_4(1, 1, -4, 1) \).

5. Decide the dependence or independence of

(a) the vectors (1, 3, 2) and (2, 1, 3) and (3, 2, 1)

(b) the vectors (1, -3, 2) and (2, 1, -3) and (-3, 2, 1).

Solution: Elimination on the matrices formed from these column vectors gives for (a):

\[
\begin{align*}
\begin{bmatrix} 1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 3 & 1 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 2 & 3 \\
0 & -5 & -7 \\
0 & -1 & -5 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 2 & 3 \\
0 & -5 & -7 \\
0 & 0 & 18/5 \end{bmatrix}
\end{align*}
\]

So this matrix has rank 3 and the columns are independent.

For (b) we get

\[
\begin{align*}
\begin{bmatrix} 1 & 2 & -3 \\
-3 & 1 & 2 \\
2 & -3 & 1 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 2 & -3 \\
0 & 7 & -7 \\
0 & -7 & 7 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 2 & -3 \\
0 & 7 & -7 \\
0 & 0 & 0 \end{bmatrix}
\end{align*}
\]

so this matrix has rank 2 and the columns are dependent.

13. Find the dimensions of these 4 spaces. Which two of the spaces are the same? (a) column space of \( A \), (b) column space of \( U \), (c) row space of \( A \), (d) row space of \( U \):

\[
A = \begin{bmatrix} 1 & 1 & 0 \\
1 & 3 & 1 \\
3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 0 \\
0 & 2 & 1 \\
0 & 0 & 0 \end{bmatrix}
\]

Solution: All dimensions are 2, (c) and (d) are the same space.

16. Find a basis for each of these subspaces of \( \mathbb{R}^4 \):
(a) All vectors whose components are equal.
(b) All vectors whose components add to zero.
(c) All vectors that are perpendicular to (1, 1, 0, 0) and (1, 0, 1, 0).
(d) The column space and the nullspace of $I$ (4 by 4)

Solution: (a) $(1, 1, 1, 1)$; (b) $(1, 0, 0, -1)$, $(0, 1, 0, -1)$, $(0, 0, 1, -1)$; (c) $(-1, 1, 0, 1)$, $(-1, 1, 1, 0)$; (d) The columns of $I$ for the column space, empty basis for the nullspace.

24. True or false (give a good reason):
(a) If the columns of a matrix are dependent, so are the rows.
(b) The column space of a 2 by 2 matrix is the same as its row space.
(c) The column space of a 2 by 2 matrix has the same dimension as its row space.
(d) The columns of a matrix are a basis for the column space.

Solution: (a) False, e.g., $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.
(b) True, both equal the rank of $A$.
(c) True, both equal the rank of $A$.
(d) False, the columns might not be linearly independent, e.g., take either example from (a) or (b).

Section 3.6

2. Find bases and dimensions for the four subspaces associated with $A$ and $B$:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}. $$

Solution: $\dim C(A) = \dim C(A^T) = \dim N(A^T) = 1$, $\dim N(A) = 2$, bases are $(1, 2)$ for $C(A)$, $(1, 2, 4)$ for $C(A^T)$, $(-2, 1, 0)$ and $(-4, 0, 1)$ for $N(A)$, and $(-2, 1)$ for $N(A^T)$.

For the matrix $B$ we have $\dim C(B) = \dim C(B^T) = 2$, $\dim N(B) = 1$, $\dim N(B^T) = 0$, bases are $(1, 2)$ and $(2, 5)$ for $C(B)$, $(1, 2, 4)$ and $(2, 5, 8)$ for $C(B^T)$, $(-4, 0, 1)$ for $N(B)$, and nothing for $N(B^T)$.

3. Find a basis for each of the four subspaces associated with $A$:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}. $$

Solution: $(1, 1, 0)$ and $(3, 4, 1)$ for $C(A)$, $(0, 1, 2, 3, 4)$ and $(0, 1, 2, 4, 6)$ for $C(A^T)$, $(1, 0, 0, 0, 0)$, $(0, -2, 1, 0, 0)$ and $(0, 0, 0, -2, 1)$ for $N(A)$, and $(1, -1, 1)$ for $N(A^T)$.

7. Suppose the 3 by 3 matrix $A$ is invertible. Write down bases for the four subspaces for $A$, and also for the 3 by 6 matrix $B = [A \ A]$.

Solution: The row and column space of $A$ are both $\mathbb{R}^3$, so a basis for those is $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. The nullspace and the left nullspace of $A$ are both zero, so the bases for those are empty. The column space of $B$ is the same as the column space of $A$, so the same basis works there, the row space has basis $(1, 0, 0, 1, 0, 0), (0, 1, 0, 0, 1, 0), (0, 0, 1, 0, 0, 1)$. The left
nullspace of $B$ is zero, so its basis is empty, and the nullspace of $B$ has basis $(-1, 0, 0, 1, 0, 0)$, $(0, -1, 0, 0, 1, 0)$ and $(0, 0, -1, 0, 0, 1)$.

13. True or false (with a reason or counterexample):
(a) If $m = n$, then the row space of $A$ equals the column space.
(b) The matrices $A$ and $-A$ share the same four subspaces.
(c) If $A$ and $B$ share the same four subspaces, then $A$ is a multiple of $B$.

Solution: (a) False, e.g., $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.
(b) True. Every column of $-A$ is a linear combination of columns of $A$ (in fact, just a multiple of one column), and vice versa, so the column spaces are the same. Every solution to $Ax = 0$ also solves $-Ax = 0$ and vice versa, so the nullspaces are the same. Since $(-A)^T = -(A^T)$, the same is true for the column space and nullspace of $A^T$ and $(-A)^T$, so the four subspaces are the same for $A$ and $-A$.
(c) False, any two $n$ by $n$ invertible matrices share the same subspaces, and they need not be multiples of each other, e.g., $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

25. True or false (with a reason or counterexample):
(a) $A$ and $A^T$ have the same number of pivots.
(b) $A$ and $A^T$ have the same left nullspace.
(c) If the row space equals the column space then $A^T = A$.
(d) If $A^T = -A$ then the row space equals the column space.

Solution: (a) True, both equal the rank of the matrix.
(b) False, any non-square matrix gives a counterexample, and it is not even true for square matrices, e.g., $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.
(c) False, any non-symmetric invertible matrix is a counterexample, e.g., $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
(d) True, because $C(A^T) = C(-A^T)$ by problem 13b, and $-A^T = A$ by assumption, so $C(-A^T) = C(A)$. 