1. True or false? Justify your answers.

(a) If a linear system of equations $Ax = b$ has more than one solution, then it has infinitely many solutions.

True, because the number of solutions to linear systems is always either 0, 1, or $\infty$.

(b) If $A$ and $B$ are two invertible $n$ by $n$ matrices, then $(AB)^{-1} = A^{-1}B^{-1}$.

False, the correct equation is $(AB)^{-1} = B^{-1}A^{-1}$. For a simple example, use $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$, so $(AB)^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$. Also, $A^{-1} = A$ and $B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, so $A^{-1}B^{-1} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \neq (AB)^{-1}$. (You don’t really need to find an example here, but it might be instructive to see how you come up with something like this. Both $A$ and $B$ are matrices corresponding to row operations. $A$ multiplies the first row with $-1$, and $B$ adds the first row to the second row. Now $AB$ first adds the first row to the second row and then multiplies the first row by $-1$, whereas $BA$ first multiplies the first row by $-1$ and then adds it to the second row. You can see that the outcome of these two is not the same, so $(AB)^{-1}$ is not the same as $(BA)^{-1} = A^{-1}B^{-1}$ either. As an exercise, try to see what the inverse matrices do as row operations.)

(c) Given an $n$ by $n$ matrix $A$, the matrix $B = A + A^T$ is always symmetric.


(d) The solutions to $Ax = b$ form a subspace.

False, unless $b = 0$ the space of solutions does not contain the zero vector, and thus can not be a subspace. (In the case $b = 0$ it is a subspace, namely the nullspace of $A$.)

(e) If $A$ is a singular $n$ by $n$ matrix, then $A^2$ is also singular.

True, for several different reasons. E.g., singular matrices have determinant zero, and determinants of products of matrices are the products of the determinants, so $|A^2| = |A|^2 = 0^2 = 0$, and so $A^2$ is singular.

(f) If $\lambda$ is not an eigenvalue of $A$, then $A - \lambda I$ is invertible.

True, the eigenvalues are the solutions of $|A - \lambda I| = 0$, so if $\lambda$ is not an eigenvalue, $|A - \lambda I| \neq 0$, and any matrix with a non-zero determinant is invertible.

2. If $u$, $v$, and $w$ are unit vectors such that $u$ is perpendicular to both $v$ and $w$, and the angle between $v$ and $w$ is $45^\circ$, find the length of the sum $u + v + w$.

Solution: This can be found by elementary geometry, but dot products make it pretty easy: $\|u + v + w\|^2 = (u + v + w) \cdot (u + v + w) = \|u\|^2 + \|v\|^2 + \|w\|^2 + 2u \cdot v + 2u \cdot w + 2v \cdot w$. Since $u$, $v$, and $w$ are unit vectors, $\|u\|^2 = \|v\|^2 = \|w\|^2 = 1$, and since $u$ is perpendicular to both $v$ and $w$, we get $u \cdot v = u \cdot w = 0$. Lastly, the angle between $v$ and $w$ being $45^\circ$ gives us $v \cdot w = \|v\|\|w\|\cos 45^\circ = \frac{1}{\sqrt{2}}$, so putting everything together we get $\|u + v + w\|^2 = 3 + \frac{2}{\sqrt{2}} = 3 + \sqrt{2}$, so $\|u + v + w\| = \sqrt{3 + \sqrt{2}}$.

3. What 3 by 3 matrix multiplies $(x, y, z)$ to give $(x + 2z, x + y - z, 2x + z)$?

Solution: $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$. 
4. Solve the following system of equations by elimination with matrices.

\[
\begin{align*}
x + 2z &= -1 \\
x + y - z &= 2 \\
2x + z &= 2
\end{align*}
\]

Solution: We already have the matrix \(A\) from problem 3, now we just have to perform Gauss elimination on the augmented matrix and then solve by back substitution:

\[
\begin{bmatrix}
1 & 0 & 2 & -1 \\
1 & 1 & -1 & 2 \\
2 & 0 & 1 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 2 & -1 \\
0 & 1 & -3 & 3 \\
0 & 0 & -3 & 3
\end{bmatrix}.
\]

Now back substitution gives \(-3z = 3\), so \(z = -1\), \(y - 3z = 3\), so \(y = 3 + 3z = 0\), and \(x + 2z = -1\), so \(x = -1 - 2z = 1\).

5. Find the determinant and inverse of

\[
A = \begin{bmatrix}
1 & 1 & 2 \\
0 & 1 & 0 \\
2 & -1 & 1
\end{bmatrix}.
\]

Solution: For the determinant we can use cofactor expansion with respect to the second row and get \(|A| = 1 \begin{vmatrix} 2 \end{vmatrix} - 1 \begin{vmatrix} 1 \end{vmatrix} = -3\). For the inverse we use Gauss-Jordan elimination:

\[
\begin{align*}
\begin{bmatrix}
1 & 1 & 2 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
2 & -1 & 1 & 0 & 0
\end{bmatrix} & \rightarrow \begin{bmatrix}
1 & 1 & 2 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & -3 & -3 & -2 & 0 & 1
\end{bmatrix} \\
& \rightarrow \begin{bmatrix}
1 & 1 & -1/3 & 2 & 2/3 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 2/3 & -1 & -1/3
\end{bmatrix} \\
& \rightarrow \begin{bmatrix}
1 & 0 & -1/3 & 1 & 2/3 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 2/3 & -1 & -1/3
\end{bmatrix},
\end{align*}
\]

so

\[
A^{-1} = \begin{bmatrix}
-1/3 & 1 & 2/3 \\
0 & 1 & 0 \\
2/3 & -1 & -1/3
\end{bmatrix}.
\]

6. Find the rank, and dimensions and bases for all four subspaces of

\[
A = \begin{bmatrix}
2 & -2 & -2 \\
1 & 1 & -1 \\
1 & -1 & -1 \\
2 & 0 & -2
\end{bmatrix}.
\]

Gauss elimination gives

\[
\begin{align*}
\begin{bmatrix}
2 & -2 & -2 \\
0 & 2 & 0 \\
0 & 0 & 0 \\
0 & 2 & 0
\end{bmatrix} & \rightarrow \begin{bmatrix}
2 & -2 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\end{align*}
\]

This show that the rank is 2, and thus \(\dim C(A) = \dim C(A^T) = 2\), \(\dim N(A) = 3 - 2 = 1\), \(\dim N(A^T) = 4 - 2 = 2\). A basis for \(C(A)\) is given by the first two columns of \(A\), i.e, \((2,1,1,2)\) and
(-2, 1, -1, 0), a basis for \( C(A^T) \) is given by the first two rows of either \( A \) or \( U \), i.e., (2, -2, -2) and (1, 1, -1), or (2, -2, -2) and (0, 2, 0). For the nullspace we have a basis formed by the one special solution \((1, 0, 1)\). The left nullspace we get by elimination on the transpose \( A^T \):

\[
\begin{bmatrix}
2 & 1 & 1 & 2 \\
-2 & 1 & -1 & 0 \\
-2 & 1 & -1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & 1 & 1 & 2 \\
0 & 2 & 0 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

This gives us a basis for \( N(A^T) \), the two special solutions \((-1/2, 0, 1, 0)\) and \((-1/2, -1, 0, 1)\).

7. With the same matrix \( A \) as in problem 6, find conditions on \((b_1, b_2, b_3, b_4)\) for solvability of \( Ax = b \). Find the complete solution for \( b = (0, 2, 0, 2) \).

Solution: Typically, one would solve this by elimination on the augmented matrix and read off the conditions on the right-hand side of the zero rows at the end. However, we can also use the result for problem 6 to answer these questions. The system is solvable whenever \( b \) is in the column space. Since the column space and the left nullspace are orthogonal complements, \( b \) is in the column space if and only if it is perpendicular to all elements in the left nullspace. This is true if and only if \( b \) is perpendicular to the two vectors \((-1/2, 0, 1, 0)\) and \((-1/2, -1, 0, 1)\) which form the basis of \( N(A^T) \). So we get the two conditions \(-b_1/2 + b_3 = 0\) and \(-b_1/2 - b_2 + b_4 = 0\) for solvability.

In order to find a particular solution, we eliminate on the augmented matrix first:

\[
\begin{bmatrix}
2 & -2 & -2 & 0 \\
1 & 1 & -1 & 2 \\
1 & -1 & -1 & 0 \\
2 & 0 & -2 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & -2 & -2 & 0 \\
0 & 2 & 0 & 2 \\
0 & 0 & 0 & 0 \\
0 & 2 & 0 & 2
\end{bmatrix}.
\]

Now we set the free variable \( x_3 = 0 \), and solve by back substitution: \( 2x_2 = 2 \), so \( x_2 = 1 \), and \( 2x_1 - 2x_2 - 2x_3 = 0 \), so \( x_1 = x_2 + x_3 = 1 \), giving the particular solution \( x_p = (1, 1, 0) \). We already know the nullspace solution \( x_n = x_3(1, 0, 1) \) from problem 6, so the complete solution is \( x = x_p + x_n = (1, 1, 0) + x_3(1, 0, 1) = (1 + x_3, 1, x_3) \).

8. Let \( A \) be an invertible 3 by 3 matrix. Find the four subspaces, rank, and dimension of the block matrices

\[
B = \begin{bmatrix} A & A \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}.
\]

Solution: The columns of \( B \) are the same as the columns of \( A \), only repeated, so the column spaces of \( A \) and \( B \) are the same. Since \( A \) is invertible, \( C(A) = C(B) = \mathbb{R}^3 \). This also shows that \( B \) has rank 3 = \( \dim C(B) = \dim C(B^T) \), and thus \( \dim N(B^T) = 3 - 3 = 0 \) and \( \dim N(B) = 6 - 3 = 3 \). Performing Gauss-Jordan elimination on an invertible matrix always produces the identity matrix, so the same elimination performed on \( B \) gives \([I \ \ I]\). A basis for the nullspace is given by the special solutions \((-1, 0, 0, 1, 0, 0), (0, -1, 0, 0, 1, 0), \) and \((0, 0, -1, 0, 0, 1)\), a basis for the row space is given by the three rows \((1, 0, 0, 1, 0, 0), (0, 1, 0, 0, 1, 0), \) and \((0, 0, 1, 0, 0, 1)\). Performing elimination on \( C \) gives \( \begin{bmatrix} A & A \\ 0 & -A \end{bmatrix} \rightarrow \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \). This shows that the rank of \( C \) is 6, and thus \( \dim C(C) = \dim C(C^T) = 6 \), and \( \dim N(C) = \dim N(C^T) = 0 \). So \( C(C) = C(C^T) = \mathbb{R}^6 \), and \( N(C) = N(C^T) = \{0\} \).

9. If \( Ax = b \) has infinitely many solutions, and \( c \) is another vector, how many solutions can \( Ax = c \) possibly have?

Solution: None at all or infinitely many. The assumptions imply that \( b \) is in the column space \( C(A) \), and that the matrix \( A \) has at least one free variable. If \( c \) is also in \( C(A) \), there is a particular
solution, and then the complete solution has at least one free parameter, so there are infinitely many solutions. If \( \mathbf{c} \) is not in \( C(\mathbf{A}) \), then \( \mathbf{A}\mathbf{x} = \mathbf{c} \) has no solutions at all.

10. Find the eigenvalues and eigenvectors of

\[
\mathbf{A} = \begin{bmatrix}
0 & 0 & -2 \\
0 & 3 & 0 \\
2 & 0 & 0
\end{bmatrix}
\]

Solution:

\[
| \mathbf{A} - \lambda I | = \begin{vmatrix}
-\lambda & 0 & -2 \\
0 & 3 - \lambda & 0 \\
2 & 0 & -\lambda
\end{vmatrix} = (3 - \lambda)(\lambda^2 + 4)
\]

has roots \( \lambda_1 = 3, \lambda_{2/3} = \pm 2i \). We get the eigenvector for \( \lambda_1 = 3 \) by elimination on \( \mathbf{A} - 3 \mathbf{I} \):

\[
\begin{bmatrix}
-3 & 0 & -2 \\
0 & 0 & 0 \\
2 & 0 & -3
\end{bmatrix} \rightarrow \begin{bmatrix}
-3 & 0 & -2 \\
0 & 0 & 0 \\
0 & 0 & -3 - 4/3
\end{bmatrix},
\]

giving the eigenvector \((0,1,0)\). For \( \lambda_2 = 2i \) we get

\[
\begin{bmatrix}
-2i & 0 & -2 \\
0 & 3 - 2i & 0 \\
2 & 0 & -2i
\end{bmatrix} \rightarrow \begin{bmatrix}
-2i & 0 & -2 \\
0 & 3 - 2i & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

giving the eigenvector \((i,0,1)\). For \( \lambda_3 = -2i \) we get the complex conjugate eigenvector \((-i,0,1)\).

11. A 2 by 2 matrix has eigenvalues 1 and 2 with corresponding eigenvectors \( \mathbf{x}_1 = (1,1) \) and \( \mathbf{x}_2 = (-1,1) \). Find \( \mathbf{A} \). (Hint: Diagonalization.)

Solution: The assumptions give us \( \mathbf{A} = \begin{bmatrix}
1 & 0 \\
0 & 2
\end{bmatrix} \) and \( \mathbf{S} = \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix} \). Then \( \mathbf{S}^{-1} = \frac{1}{2} \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} \), so

\[
\mathbf{A} = \mathbf{S} \mathbf{\Lambda} \mathbf{S}^{-1} = \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
1/2 & 1/2
\end{bmatrix} = \begin{bmatrix}
1 & -1 \\
1/2 & 1/2
\end{bmatrix} = \begin{bmatrix}
3/2 & -1/2 \\
-1/2 & 3/2
\end{bmatrix}
\]