17.1 Green’s Theorem

Lukas Geyer

Montana State University

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Fundamental Theorems of Vector Analysis

- **Green’s Theorem** \( \oint_{\partial D} F \cdot ds = \iint_{D} \text{curl} \ F \ dA \)
  - \( D \) plane domain, \( F = \langle P, Q \rangle \)
  - \( \text{curl} \langle P, Q \rangle = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \)

- **Stokes’ Theorem** \( \oint_{\partial S} F \cdot ds = \iint_{S} \text{curl} \ F \cdot dS \)
  - \( S \) surface in space, \( F = \langle P, Q, R \rangle \)
  - \( \text{curl} \langle P, Q, R \rangle = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle \)

- **Divergence Theorem** \( \oiint_{\partial W} F \cdot dS = \iiint_{W} \text{div} \ F \ dV \)
  - \( W \) region in space, \( F = \langle P, Q, R \rangle \)
  - \( \text{div} \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \)
Green’s Theorem

**Theorem**

\[
\oint_{\partial D} F_1 \, dx + F_2 \, dy = \iint_{D} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA
\]

**Remarks**

- **D** is a domain bounded by a finite number of simple closed smooth curves.
- **\( \partial D \)** is parametrized in the mathematically positive direction, i.e., such that **D** always lies to the left of **\( \partial D \)**.
- **F = \langle F_1, F_2 \rangle** is a smooth vector field.
- \( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \) is the **curl** of **F**. Physically, the curl is *circulation per unit of enclosed area*. 
Example 1

Verify Green’s Theorem for the line integral along the unit circle $C$, oriented counterclockwise:

\[
\oint_C y \, dx + xy \, dy
\]

Direct Way

\[
x = \cos \theta, \quad y = \sin \theta, \quad dx = -\sin \theta \, d\theta, \quad dy = \cos \theta \, d\theta
\]

\[
\oint_C y \, dx + xy \, dy = \int_0^{2\pi} (\sin \theta)(-\sin \theta) + (\cos \theta \sin \theta)(\cos \theta) \, d\theta
\]

\[
= \int_0^{2\pi} -\sin^2 \theta + \cos^2 \theta \sin \theta \, d\theta
\]
Example II

Example

Verify Green's Theorem for the line integral along the unit circle $C$, oriented counterclockwise:

$$\int_C y \, dx + xy \, dy$$

Direct Way

$$\oint_C y \, dx + xy \, dy = \int_0^{2\pi} -\sin^2 \theta + \cos^2 \theta \sin \theta \, d\theta$$

$$= \left[ -\frac{\theta}{2} + \frac{\sin 2\theta}{4} - \frac{\cos^3 \theta}{3} \right]_0^{2\pi} = -\pi$$
Example III

Example

Verify Green’s Theorem for the line integral along the unit circle $C$, oriented counterclockwise:

$$\int_C y \, dx + xy \, dy$$

Green’s Way

$$F_1 = y, \quad F_2 = xy, \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = y - 1$$

$$\iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA = \iint_D (y - 1) \, dA = \int_0^1 \int_0^{2\pi} (r \sin \theta - 1) r \, d\theta \, dr$$

$$= \int_0^1 [-r^2 \cos \theta - r\theta]_{\theta = 0}^{\theta = 2\pi} \, dr = \int_0^1 (-2\pi r) \, dr = -\pi$$
Calculating Area

**Theorem**

\[ \text{area}(\mathcal{D}) = \frac{1}{2} \int_{\partial \mathcal{D}} x \, dy - y \, dx \]

**Proof.**

\[ F_1 = -y, \quad F_2 = x, \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 - (-1) = 2, \]

\[ \frac{1}{2} \int_{\partial \mathcal{D}} x \, dy - y \, dx = \frac{1}{2} \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dA = \frac{1}{2} \iint_D 2 \, dA = \text{area}(\mathcal{D}). \]
Example

Find the area of the quadrilateral with vertices \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) and \((x_4, y_4)\), using Green’s Theorem.

Parametrizing one side
For \(0 \leq t \leq 1\),

\[
\mathbf{c}(t) = \langle x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1) \rangle
\]

\[
dx = (x_2 - x_1)dt
\]

\[
dy = (y_2 - y_1)dt
\]
Area of a Quadrilateral II

Example

Find the area of the quadrilateral with vertices \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) and \((x_4, y_4)\), using Green’s Theorem.

Integrating over one side

\[ c(t) = \langle x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1) \rangle, \quad 0 \leq t \leq 1 \]

\[ dx = (x_2 - x_1)dt, \quad dy = (y_2 - y_1)dt \]

\[
\begin{align*}
  x \ dy & = (x_1 + t(x_2 - x_1))(y_2 - y_1)dt \\
  y \ dx & = (y_1 + t(y_2 - y_1))(x_2 - x_1)dt \\
  x \ dy - y \ dx & = (x_1(y_2 - y_1) - y_1(x_2 - x_1))dt \\
  & = (x_1y_2 - x_2y_1)dt
\end{align*}
\]
Area of a Quadrilateral III

**Example**

Find the area of the quadrilateral with vertices \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) and \((x_4, y_4)\), using Green’s Theorem.

**Integrating over one side**

\[
xdy - ydx = (x_1y_2 - x_2y_1)dt
\]

\[
\frac{1}{2} \int_{C_1} xdy - ydx = \frac{1}{2} \int_0^1 (x_1y_2 - x_2y_1)dt = \frac{x_1y_2 - x_2y_1}{2}
\]
Area of a Quadrilateral IV

Example

Find the area of the quadrilateral with vertices \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) and \((x_4, y_4)\), using Green’s Theorem.

Integrating over one side

\[
\frac{1}{2} \int_{C_1} x \, dy - y \, dx = \frac{x_1 y_2 - x_2 y_1}{2}
\]

Integrating over all sides

\[
\text{area}(D) = \frac{(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_4 - x_4 y_3) + (x_4 y_1 - x_1 y_4)}{2}
\]
The 2-D Divergence Theorem I

Definition
If $C$ is a closed curve, $\mathbf{n}$ the outward-pointing normal vector, and $\mathbf{F} = \langle P, Q \rangle$, then the flux of $\mathbf{F}$ across $C$ is $\oint_C (\mathbf{F} \cdot \mathbf{n}) \, ds$.

Remark
If the tangent vector to the curve $C$ is $\langle x'(t), y'(t) \rangle$, the outward-pointing normal vector is $\langle y'(t), -x'(t) \rangle$, so the flux is

$$\oint_C \langle P, Q \rangle \cdot \langle dy, -dx \rangle = \oint_C P \, dy - Q \, dx$$

Theorem
The flux of $\mathbf{F}$ across $C$ is

$$\iint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dA$$
The 2-D Divergence Theorem II

**Theorem**

The flux of $\mathbf{F}$ across $\mathcal{C}$ is

$$
\int\int_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA
$$

**Definition**

The quantity $\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is the divergence of $\mathbf{F} = \langle P, Q \rangle$.

**Theorem**

$$
\oint_{\mathcal{C}} (\mathbf{F} \cdot \mathbf{n}) ds = \int\int_D \text{div } F dA
$$
The 2-D Divergence Theorem Proof

**Theorem**

\[ \text{Flux} = \oint_C P \, dy - Q \, dx = \int\int_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dA \]

**Proof.**

Using Green’s Theorem,

\[
\oint_C P \, dy - Q \, dx = \oint_C -Q \, dx + P \, dy = \int\int_D \left( \frac{\partial}{\partial x} P - \frac{\partial}{\partial y} (-Q) \right) \, dA = \int\int_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dA
\]
2-D Divergence Example

Example

Find the flux of \( \mathbf{F}(x, y) = (2x + 2xy + y^2, x + y - y^2) \) across the circle \( x^2 + y^2 = 4 \).

Using the 2-D Divergence Theorem

\[
\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 2 + 2y + 1 - 2y = 3
\]

So

\[
\text{Flux} = \iint_D \text{div } \mathbf{F} \, dA = \iint_D 3 \, dA = 3 \text{ area}(D) = 12\pi
\]