17.2 Stokes’ Theorem

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Fundamental Theorems of Vector Analysis

- **Green’s Theorem** \( \oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D \text{curl} \mathbf{F} \, dA \)
  - \( D \) plane domain, \( \mathbf{F} = \langle P, Q \rangle \)
  - \( \text{curl} \langle P, Q \rangle = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \)

- **Stokes’ Theorem** \( \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S} \)
  - \( S \) surface in space, \( \mathbf{F} = \langle P, Q, R \rangle \)
  - \( \text{curl} \langle P, Q, R \rangle = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle \)

- **Divergence Theorem** \( \iiint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iiint_W \text{div} \mathbf{F} \, dV \)
  - \( W \) region in space, \( \mathbf{F} = \langle P, Q, R \rangle \)
  - \( \text{div} \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \)
Stokes’ Theorem

**Theorem**

\[ \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_{S} \text{curl} \mathbf{F} \cdot d\mathbf{S} \]

**Remarks**

- \( S \) is an oriented surface in space.
- \( \partial S \) has the **boundary orientation**: If a unit normal vector is walking along \( \partial S \), the surface \( S \) is to its left.
- \( \mathbf{F} = \langle F_1, F_2, F_3 \rangle \) is a smooth vector field.
- \( \text{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle F_1, F_2, F_3 \rangle \)
  
  \[ = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle \]
Example

Verify Stokes’ Theorem for the surface $z = x^2 + y^2$, $0 \leq z \leq 4$, with upward pointing normal vector and $\mathbf{F} = \langle -2y, 3x, z \rangle$.

Computing the line integral

The boundary $\partial S$ is the circle $x^2 + y^2 = 4$ in the $z = 4$ plane. Standard parametrization is

$$\mathbf{c}(\theta) = (2 \cos \theta, 2 \sin \theta, 4), \quad \mathbf{c}'(\theta) = \langle -2 \sin \theta, 2 \cos \theta, 0 \rangle.$$ 

The orientation is correct, too: An upward-pointing normal vector moving around $\partial S$ counterclockwise sees $S$ to its left.
Example II

Example

Verify Stokes’ Theorem for the surface $z = x^2 + y^2$, $0 \leq z \leq 4$, with upward pointing normal vector and $\mathbf{F} = \langle -2y, 3x, z \rangle$.

Computing the line integral

$$\mathbf{c}(\theta) = (2 \cos \theta, 2 \sin \theta, 4), \quad \mathbf{c}'(\theta) = \langle -2 \sin \theta, 2 \cos \theta, 0 \rangle.$$

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \langle -4 \sin \theta, 6 \cos \theta, 4 \rangle \cdot \langle -2 \sin \theta, 2 \cos \theta, 0 \rangle d\theta$$

$$= \int_0^{2\pi} 8 \sin^2 \theta + 12 \cos^2 \theta \, d\theta = 8\pi + 12\pi = 20\pi.$$
Example III

Example

Verify Stokes’ Theorem for the surface \( z = x^2 + y^2 \), \( 0 \leq z \leq 4 \), with upward pointing normal vector and \( F = \langle -2y, 3x, z \rangle \).

Computing the surface integral

We can use \( x \) and \( y \) as parameters over the disk \( x^2 + y^2 \leq 4 \). Then

\[
G(x, y) = (x, y, x^2 + y^2),
\]

\[
T_x = \langle 1, 0, 2x \rangle, \quad T_y = \langle 0, 1, 2y \rangle \quad n = T_x \times T_y = \langle -2x, -2y, 1 \rangle.
\]

Orientation is correct because the third component of \( n \) is positive, so \( n \) is pointing up.
Example IV

Example

Verify Stokes’ Theorem for the surface \( z = x^2 + y^2, \ 0 \leq z \leq 4 \), with upward pointing normal vector and \( \mathbf{F} = \langle -2y, 3x, z \rangle \).

Computing the surface integral

\[
G(x, y) = (x, y, x^2 + y^2), \quad x^2 + y^2 \leq 4, \quad \mathbf{n} = \langle -2x, -2y, 1 \rangle.
\]

\[
\text{curl} \ \mathbf{F} = \left\langle \frac{\partial \mathbf{F}_3}{\partial y} - \frac{\partial \mathbf{F}_2}{\partial z}, \frac{\partial \mathbf{F}_1}{\partial z} - \frac{\partial \mathbf{F}_3}{\partial x}, \frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y} \right\rangle
= \langle 0 - 0, 0 - 0, 3 - (-2) \rangle = \langle 0, 0, 5 \rangle
\]
Example V

Verify Stokes’ Theorem for the surface $z = x^2 + y^2$, $0 \leq z \leq 4$, with upward pointing normal vector and $\mathbf{F} = \langle -2y, 3x, z \rangle$.

Computing the surface integral

$$G(x, y) = (x, y, x^2 + y^2), \quad x^2 + y^2 \leq 4, \quad \mathbf{n} = \langle -2x, -2y, 1 \rangle.$$  

$\text{curl} \mathbf{F} = \langle 0, 0, 5 \rangle$

$$\int_S \text{curl} \mathbf{F} \cdot d\mathbf{S} = \int \int_D \langle 0, 0, 5 \rangle \cdot \langle -2x, -2y, 1 \rangle \, dA$$

$$= \int \int_D 5 \, dA = 5 \text{area}(D) = 5 \cdot 4\pi = 20\pi.$$