Final Review Problems Key, M 273, Fall 2011

1. Show that the equation \(\cot \phi = 2 \cos \theta + \sin \theta\) in spherical coordinates describes a plane through the origin, and find a normal vector to this plane.

Multiply by \(\rho \sin \phi\) to get \(\rho \cos \phi = 2 \rho \sin \phi \cos \theta + \rho \sin \phi \sin \theta\), i.e., \(z = 2x + y\). A normal vector is \((2, 1, -1)\).

2. Find the length of the path \(r(t) = (\sin 2t, \cos 2t, 3t - 1)\) for \(1 \leq t \leq 3\).

\[
L = \int_{1}^{3} \|r'(t)\| \, dt = \int_{1}^{3} \sqrt{13} \, dt = 2\sqrt{13}
\]

3. A force \(\mathbf{F} = (12t + 4, 8 - 24t)\) (Newtons) acts on a 2-kg mass. Find the position of the mass at \(t = 2\) (seconds) if it is located at \((4, 6)\) at \(t = 0\) and has initial velocity \((2, 3)\) (m/s).

\[\mathbf{F} = m \mathbf{a}, \text{ so } \mathbf{a}(t) = (6t + 2, 4 - 12t), \mathbf{v}(t) = (2 + 2t + 3t^2, 3 + 4t - 6t^2), r(t) = (4 + 2t + t^3, 6 + 3t + 2t^2 - 2t^3), r(2) = (20, 4).\]

4. Find an equation of the tangent plane at \(P = (0, 3, -1)\) to the surface with equation \(ze^x + e^{z+1} = xy + y - 3\).

The equation is a level surface of \(\mathbf{F}(x, y, z) = ze^x + e^{z+1} - xy - y\), whose gradient is \(\nabla \mathbf{F}(x, y, z) = \langle ze^x - y, -x - 1, e^x + e^{z+1}\rangle\), so the normal vector is \(\mathbf{n} = \nabla \mathbf{F}(0, 3, -1) = \langle -4, -1, 2\rangle\), and the equation of the plane is \(-4x - y + 2z = -4\cdot 0 - 1\cdot 3 + 2\cdot (-1) = -5\).

5. Find the minimum and maximum values of \(f(x, y, z) = x - z\) on the intersection of the cylinders \(x^2 + y^2 = 1\) and \(x^2 + z^2 = 1\).

Lagrange multipliers give equations \(1 = 2\lambda x + 2\mu x, 0 = 2\lambda y, \text{ and } -1 = 2\mu z\). The second equation implies \(\lambda = 0\) or \(y = 0\). The case \(y = 0\) leads to \(x = \pm 1\) and \(z = 0\), so \(f(x, y, z) = \pm 1\). The case \(\lambda = 0\) leads to \(2\mu x = 1 = -2\mu z\). Dividing by \(2\mu\) gives \(x = -z\). Using \(x^2 + z^2 = 1\) we have the two solutions \(x = 1/\sqrt{2}\) and \(z = -1/\sqrt{2}\), or \(x = -1/\sqrt{2}\) and \(z = 1/\sqrt{2}\). In either case \(y = \pm 1/\sqrt{2}\), and the two possible values for \(f(x, y, z)\) are \(\pm 2/\sqrt{2} = \pm \sqrt{2}\). So the maximum is \(\sqrt{2}\), the minimum is \(-\sqrt{2}\).

6. Find the double integral of \(f(x, y) = x^3y\) over the region between the curves \(y = x^2\) and \(y = x(1 - x)\).

\[
\iint_{D} x^3y \, dA = \int_{0}^{1/2} \int_{x^2}^{x(1-x)} x^3y \, dy \, dx = \int_{0}^{1/2} x^3 \left[ \frac{y^2}{2} \right]_{y=x^2}^{y=x(1-x)} \, dx
\]
\[
= \int_{0}^{1/2} x^3 \left( \frac{x-x^2}{2} - x^4 \right) \, dx = \int_{0}^{1/2} x^5 \left( \frac{x^2}{2} - 2x^6 \right) \, dx
\]
\[
= \frac{1}{2} \left( \frac{5}{6} - \frac{1}{2} - \frac{2}{7} + \frac{1}{2^7} \right) = \frac{1}{5376}
\]
7. Use cylindrical coordinates to find the mass of the solid bounded by 
\[ z = 8 - x^2 - y^2 \]
and 
\[ z = x^2 + y^2 \], assuming a mass density of 
\[ f(x, y, z) = (x^2 + y^2)^{1/2} \].

\[
m = \iiint_V f(x, y, z) \, dV = \iiint_V r \, dV = \int_0^2 \int_0^{2\pi} \int_r^8 r^2 \, dz \, d\theta \, dr
\]
\[
= 2\pi \int_0^2 (8r^2 - 2r^4) \, dr = 2\pi \left( \frac{8}{3} \cdot 8 - \frac{2}{5} \cdot 32 \right) = \frac{256\pi}{15}.
\]

8. Calculate the work required to move an object from 
\[ P = (1, 1, 1) \] to \[ Q = (3, -4, -2) \] against the force field 
\[ \mathbf{F}(x, y, z) = -12r^{-4} \langle x, y, z \rangle \] (distance in meters, force in Newtons),
where \( r = \sqrt{x^2 + y^2 + z^2} \). Hint: Find a potential function for \( \mathbf{F} \).

Potential is 
\[ V(x, y, z) = 6(x^2 + y^2 + z^2)^{-1} = 6r^{-2} \], so the work required (where \( C \) is any path from \( P \) to \( Q \)) is
\[ W = -\int_C \mathbf{F} \cdot d\mathbf{s} = -(V(Q) - V(P)) = V(P) - V(Q) = V(1, 1, 1) - V(3, -4, -2) = \frac{6}{3} - \frac{6}{25} = \frac{52}{25}.
\]

9. Find the flow rate of a fluid with velocity field 
\[ \mathbf{v} = \langle 2x, y, xy \rangle \] m/s across the part of the cylinder 
\[ x^2 + y^2 = 9 \] where \( x \geq 0, y \geq 0, \) and \( 0 \leq z \leq 4 \) (distance in meters).

Parametrization \( G(\theta, z) = (3\cos \theta, 3\sin \theta, z) \), with \( 0 \leq \theta \leq \pi/2 \) and \( 0 \leq z \leq 4 \). Then
\[ \mathbf{T}_\theta = \langle -3\sin \theta, 3\cos \theta, 0 \rangle, \quad \mathbf{T}_z = \langle 0, 0, 1 \rangle, \quad \mathbf{n} = \mathbf{T}_\theta \times \mathbf{T}_z = \langle 3\cos \theta, 3\sin \theta, 0 \rangle \] (outward-pointing), and

Flow rate
\[
= \int_0^{\pi/2} \int_0^4 (6\cos \theta, 3\sin \theta, 9\cos \theta \sin \theta) \cdot (3\cos \theta, 3\sin \theta, 0) \, dz \, d\theta
\]
\[
= \int_0^{\pi/2} \int_0^4 (18\cos^2 \theta + 9\sin^2 \theta) \, dz \, d\theta = 4 \left( 18 \cdot \frac{\pi}{4} + 9 \cdot \frac{\pi}{4} \right) = 27\pi
\]

10. Use Green’s Theorem to evaluate \( \oint_C xy \, dy - y^2 \, dx \), where \( C \) is the unit circle in counterclockwise orientation.

\[
\oint_C xy \, dy - y^2 \, dx = \iint_D \left( \frac{\partial}{\partial x} xy - \frac{\partial}{\partial y} (-y^2) \right) \, dA
\]
\[
= \iint_D 3y \, dA = 0
\]

by symmetry. (The integral is the first moment \( M_x \) of a unit disk with constant mass density \( \rho = 3 \).)