with some errata

0	1	2	3	4	5	6	7	8	9	10	
10	10	15	10	10	10	10	15	15	10	10	125

Math 333 Final Exam (16 Dec 2011)

#1 $\{e_1, \ldots, e_n\}$

Errata: #6 Assume A is 3×3

#8 c) $T - 2I$ should perhaps be $T - 2Id$ and rank(S) = dim range(S)

#9 b) identify $\tilde{u}, \tilde{v}$

#10 draw vector u, v.

Name:

Show all work (unless instructed otherwise). Good Luck!

0.[10pts] Circle **True** or **False** without explanation. Below V is a finite dimensional vector space.

(**T** or **F**) V is isomorphic to $\mathbf{R}^n$ for some n.

(**T** or **F**) If V has two **orthogonal** subspaces of dimension 3 then $\dim(V) \geq 6$.

(**T** or **F**) For T linear, range$(T) \subset \ker(T)$ is equivalent to $T \circ T = 0$.

(**T** or **F**) The number of (non-zero) singular values of A is the rank of A.

(**T** or **F**) If the operator norm $\|A\| < 1$, then $\lim_{n\to\infty} \|A^n\| = 0$

1.[10pts] Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_1\}$ be the standard basis of $\mathbf{R}^n$ and $\|x\|$ be a norm of $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

e_n

a) Show that $\|x\| \leq |x_1| \cdot \|\mathbf{e}_1\| + \ldots + |x_n| \cdot \|\mathbf{e}_n\|$ for all $x \in \mathbf{R}^n$. (Justify steps.)

$$\|x\| = \|x_1 e_1 + \ldots + x_n e_n\|$$

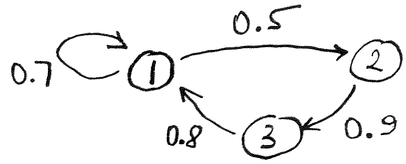
$$\leq \|x_1 e_1\| + \ldots + \|x_n e_n\| \quad \text{(by the triangle inequality)}$$

$$= |x_1|\,\|e_1\| + \ldots + |x_n|\,\|e_n\| \quad \text{(by the } \|av\| = |a|\,\|v\| \text{ axiom)}$$

b) Show $\|x\|_E \leq \|x\|_s$ for all $x \in \mathbf{R}^n$. (Here $\|x\|_E = \sqrt{\sum_i x_i^2}$ and $\|x\|_s = \sum_i |x_i|$.)

Using a), $\|x\|_E \leq |x_1|\cdot 1 + \ldots + |x_n|\cdot 1$ since $\|e_i\|_E = 1$

2.[15pts] The following weighted digraph describes the annual flow of wealth between three agents.



a) Write out the adjacency matrix A and explain in terms of the graph why $A^N > 0$ for some N. What is the smallest such N?

$$A = \begin{bmatrix} 0.7 & 0.5 & 0 \\ 0 & 0 & 0.9 \\ 0.8 & 0 & 0 \end{bmatrix}$$

Since one can get from any vertex to any other vertex in "four steps we have

$A^4 > 0$

b) Given the approximate Perron eigenvalue and eigenvectors for A and A^T:

$$\lambda \approx 1.036, \quad v_A \approx \begin{bmatrix} 0.70 \\ 0.47 \\ 0.54 \end{bmatrix} \quad v_{A^T} \approx \begin{bmatrix} 0.84 \\ 0.41 \\ 0.35 \end{bmatrix}$$

← Think which one to use, V_A or V_{A^T}? What's the "transfer matrix"

what is the asymptotic ratio of wealth between agents 2 and 1 ?

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 0.7x_1 + 0.8x_3 \\ 0.5x_1 \\ 0.9x_2 \end{pmatrix} = \begin{bmatrix} 0.7 & 0 & 0.8 \\ 0.5 & 0 & 0 \\ 0 & 0.9 & 0 \end{bmatrix} = A^T$$

Hence the asymptotic ratio is dictated by the eigenvector of A^T.

It is $\frac{0.41}{0.84} = \frac{41}{84}$.

c) What is the annual percentage rate of growth of this small *economy*?

$\lambda \approx 1.036$ means 3.6% growth rate

3.[10pts] Let $\|A\|$ be a matrix norm of a square matrix A compatible with a vector norm $\|x\|$. Suppose there is x such that

$$\|x\| = 1 \quad \text{and} \quad \|Ax\| = 2 \quad \text{and} \quad \|A^2x\| = 9.$$

Give a lower bound on $\|A\|$. That is deduce an inequality $\|A\| \geq \square$ where $\square$ is some number, **the larger the better**. (<u>Note:</u> That $\|A\| \geq 2$ is easy. $\|A\| \geq 3$ is harder. But try to do even better.)

$\|A^2x\| = \|A\,Ax\| \leq \|A\|\,\|Ax\|$ hence

$9 \leq \|A\|\,2$ yielding $\|A\| \geq 9/2 = 4.5$

4.[10pts] Let $T : V \to V$ be a linear transformation. Suppose $\mathcal{C} := \{Tv_1, \ldots, Tv_n\}$ is a basis of V. Prove that $\mathcal{B} := \{v_1, \ldots, v_n\}$ is a basis of V as well.

It suffices to prove that $\mathcal{B}$ is independent since, having $n = \dim(V)$ elements, it is then automatically spanning.

To see independence, suppose $c_1v_1 + \ldots + c_nv_n = 0$ / applying T

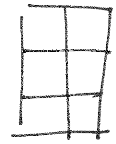
Then $T(c_1v_1 + \ldots + c_nv_n) = T(0)$

(so by linearity of T) $\rightarrow$ $T(c_1v_1) + \ldots + T(c_nv_n) = 0$

$\searrow$ $c_1Tv_1 + \ldots + c_nTv_n = 0$

From lin indep of $\mathcal{C}$, $c_1 = c_2 = \ldots = c_n = 0$

Hence $\mathcal{B}$ is independent.

$A =$ 

5.[10pts] Suppose that $A = QR$ is the QR-decomposition of a 3×2 matrix (with independent columns).

4pts a) Show that A and R have the same singular values.

For A, Singular values are the (non-zero) eigenvalues of A^TA.

But $A^TA = (QR)^TQR = R^T\underbrace{Q^TQ}_{I}R = R^TR$.

So they are the singular values of R as well.

2pts b) What geometric figure is the image of the unit circle in $\mathbf{R}^{2}$ under T_A? should be 3 my bad so take off max -1pt for this.

ellipse

4pts c) Find the singular values if $R = \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{bmatrix}$.

$R^TR = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$ trace $= 4$, det $= 3 - 2 = 1$

$\lambda^2 - 4\lambda + 1 = 0$ yields $\lambda = \dfrac{4 \pm \sqrt{4^2 - 4}}{2} = \dfrac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$

$\sigma_1 = \sqrt{2+\sqrt{3}}$, $\sigma_2 = \sqrt{2-\sqrt{3}}$

6.[10pts] Suppose you know that $y = [1, 2, 3]^T$ solves the homogeneous equation $A^Ty = 0$ and all other solutions are just scalar multiples of that one. Assume A is 3×3.

a) Explain why there is no solution to $Ax = b$ for $b = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

We know that $N(A^T) = \text{span}([1, 2, 3]^T)$

Solvability of $Ax = b$ takes place exactly when $b \in C(A)$.

That is $b \in N(A^T)^\perp$ i.e. $[1, 2, 3]^T b = 0$

However, $[1, 2, 3]\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 1 - 2 + 3 \neq 0$

b) Tweak one entry in the b above so that a solution x exists. Is x unique, or there are infinitely many solutions x? 3pts

Take $b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ so that $1 \cdot 1 + 2 \cdot (-2) + 1 \cdot 3 = 0$

Then $b \in N(A^T)^\perp = C(A)$ and $Ax = b$ is solvable.

The general solution is $X = X_{\text{particular}} + X_n$

where $X_n \in N(A)$. From a), $1 = \dim N(A^T) = 3 - r$ so $r = 2$.

Thus $\dim N(A) = 3 - r = 1$ so there are ∞-many X_n

8pts

7.[15pts] Suppose that $u, v \in \mathbf{R}^n$ are some **fixed** vectors.

a) Prove that the set $W := \{A \in M_{n \times n} : u^T A v = 0\}$ is a subspace (of $M_{n\times n}$).

Suppose $A, \tilde{A} \in W$, i.e., $u^T A v \overset{①}{=} 0$ and $u^T \tilde{A} v \overset{②}{=} 0$.

Then $u^T(A+\tilde{A})v = (u^T A + u^T \tilde{A})v = u^T A v + u^T \tilde{A} v \underset{\text{by ① and ②}}{=} 0 + 0 = 0$

This shows that $A, \tilde{A} \in W$.

Also, if c is a scalar, then

$u^T(cA)v = c\, u^T A v \underset{\text{via ①}}{=} c \cdot 0 = 0$ so $cA \in W$.

This shows that W is closed under $+$ and scalar mult.
As it is also non-empty (since $A = \mathbf{0} \in W$), it is a linear space.

7pts

b) Find a basis for W when $n = 2$ and $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(Note: The basis will consist of several 2×2 matrices.)

$$[1, 1] \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$$[a+c \quad b+d] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$$a + c - (b+d) = 0 \quad \text{so}$$

$$a = b + d - c \qquad b, d, c \text{ free}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b+d-c & b \\ c & d \end{bmatrix} = b \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A Basis is given by these three matrices

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

8.[15pts] Consider $T : \mathcal{P}_2 \to \mathcal{P}_2$ given by $T(p(x)) = p(x+1) + p(x-1)$.

5pts a) Find the matrix $[T]$ of T with respect to the standard basis $\{1, x, x^2\}$.

$1 \mapsto 1 + 1 = 2$

$x \mapsto x+1+x-1 = 2x$

$x^2 \mapsto (x+1)^2 + (x-1)^2 = 2x^2+2$

so $[T] = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

3pts b) Verify that T is invertible. (You can use a) but explain how.)

$\det [T] = 2 \cdot 2 \cdot 2 = 8 \neq 0$ so $[T]$ invertible

so T invertible

3pts c) $S = T - 2I$ is not invertible. What is the rank(S)? (You can use a).)

$[S] = [T] - [2I] = [T] - 2I = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

is singular with rank $= r = 1$

4pts d) Find a basis of the kernel of S.

Visibly the null space of $[S]$ is span$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

Hence $\ker(S) = \text{Span}\{1, x\}$

(because $[1] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $[x] = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$).

9.[10pts] For $u, v \in \mathbf{R}^2$, define $\langle u|v\rangle = 5u_1v_1 - u_1v_2 - u_2v_1 + 5u_2v_2$.

a) Verify that $\langle u|u\rangle \geq 0$ for all $u \in \mathbf{R}^2$.

$$\langle u|u\rangle = 5u_1^2 - 2u_1u_2 + 5u_2^2$$
$$= u_1^2 - 2u_1u_2 + u_2^2 + 4u_1^2 + 4u_2^2$$
$$= (u_1 - u_2)^2 + 4u_1^2 + 4u_2^2 \geq 0$$

since squares are non-negative always.

b) ~~Find the basis $\mathcal{B}$ so that the inner product can be rewritten as~~

$$(*) \qquad \langle u|v\rangle = 4\tilde{u}_1\tilde{v}_1 + 6\tilde{u}_2\tilde{v}_2. \qquad \text{where } \tilde{u} = [u]_{\mathcal{B}} \text{ and } \tilde{v} = [v]_{\mathcal{B}}$$

(Note: The eigenvalues you need are visible above. Do not compute them!)

$A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$ makes $\langle u|v\rangle = \langle u|v\rangle_A = u^T A v$

If we diagonalize $A = Q\Lambda Q^T = Q\begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}Q^T$

based on $(*)$

then $u^T A v = u^T Q \Lambda Q^T v = (Q^T u)\Lambda(Q^T v)$

so $\tilde{u} = Q^T u$, $\tilde{v} = Q^T v$.

We want a basis $\mathcal{B}$ so that $[u]_{\mathcal{B}} = [I]_{\mathcal{B}\leftarrow st} \cdot u = Q^T u$

Hence $[I]_{\mathcal{B}\leftarrow st} = Q^T$ so $[I]_{st\leftarrow\mathcal{B}} = Q$.

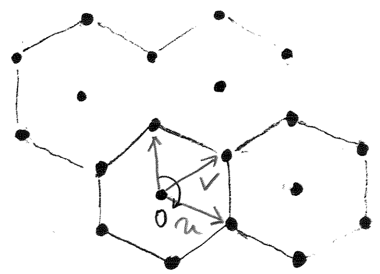
So the basis vectors are the normalized eigenvectors of A.

For $\lambda_1 = 4$, $A - 4I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ with $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so $q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = 6$ $A - 6I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ with $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so $q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

<u>Answer</u> $\mathcal{B} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \right\}$

10.[10pts] Let $R : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation by 60^o clockwise. Use the fact that it preserves the hexagonal lattice to find a basis $\mathcal{B}$ in $\mathbf{R}^2$ such that the matrix $[R]_{\mathcal{B}\leftarrow\mathcal{B}}$ has **integer entries**. Write out $[R]_{\mathcal{B}\leftarrow\mathcal{B}}$.



4 pts for u, v

6 pts for $[R]_{\mathcal{B}\leftarrow\mathcal{B}}$

Take u and v as depicted and $\mathcal{B} = \{u, v\}$

Look $Ru = v$ and $Rv = -u + v$

Hence $[R]_{\mathcal{B}\leftarrow\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$