0	1	2	3	4	5	6	7	8	
10	15	15	10	10	15	10	10	20	115

Math 333 Second Exam (22 Nov 2011)

Name:

Show all work (unless instructed otherwise). Good Luck!

0.[10pts] Circle **True** or **False** without explanation:

(**T** or **F**) The Frobenius matrix norm is an operator norm.

(**T** or **F**) The (non-zero) singular values of A and A^T are the same.

(**T** or **F**) There are **only** finitely many different inner products on $\mathbf{R}^2$.

(**T** or **F**) $A = \sigma_1 u_1 v_1^T + \ldots + \sigma_r u_r v_r^T$ is a way to express the SVD of A.

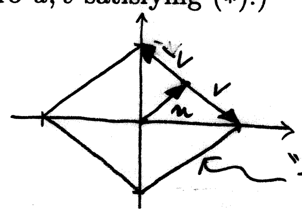
(**T** or **F**) $\left(\sum_i x_i y_i\right)^2 \le \left(\sum_i x_i^2\right)\left(\sum_i y_i^2\right)$ would not surprise Cauchy or Schwarz.

1.[15pts] Consider the question of whether it is possible that u, v are non-zero vectors and

$$(*) \qquad \|u\| = 1, \quad \|u+v\| = 1, \quad \|u-v\| = 1.$$

a) Show that the answer is "Yes" when using the *sum norm*, $\|x\| = |x_1| + |x_2|$. (You have to give a concrete example of non-zero u, v satisfying $(*)$.)

Take $u = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$,



GEOMETRIC IDEA:

"the unit circle" given by $\|x\| = 1$.

b) Prove that if the norm comes from an **inner product**, then the answer is "No". (Hint: Square $(*)$ and then write it out in terms of $\langle\,,\,\rangle$. Simplify to get $\|v\|^2 = 0$.)

$(*)$ becomes: $\|u\|^2 = 1$, $\|u+v\|^2 = \langle u+v \mid u+v\rangle = \|u\|^2 + 2\langle u \mid v\rangle + \|v\|^2 = 1$,

$\|u-v\|^2 = \langle u-v \mid u-v\rangle = \|u\|^2 - 2\langle u \mid v\rangle + \|v\|^2 = 1$.

By adding the last two equations we get:

$2\|u\|^2 + 2\|v\|^2 = 2$, so (using that $\|u\|^2 = 1$),

$2\|v\|^2 = 0$

$\|v\|^2 = 0$ so $v = 0$ contrary to the requirement that both u and v are not zero.

2.[15pts] For $u, v \in \mathbf{R}^2$, define $\langle u, v\rangle = 4u_1v_1 - 2u_1v_2 - 2u_2v_1 + 7u_2v_2$

a) Verify that $\langle u, u\rangle \geq 0$ for all $u \in \mathbf{R}^2$. (It is easiest to "complete the square".)

$$\langle u|u\rangle = 4u_1^2 - 4u_1u_2 + 7u_2^2 = (2u_1 - u_2)^2 + 6u_2^2 \geq 0$$

as a sum of two squares

NOTE: Another way is to see that $\langle u|u\rangle = u^TAu$ for $A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$, which is symmetric and positive definite since $4>0$ and $\mathrm{Det} = 4\cdot 7 - 2\cdot 2 > 0$.

Review 221 material on positive definite matrices.

b) Write out the matrix A such that $\langle u, v\rangle = u^TAv$.

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$

c) As explained in class, in some basis $\mathcal{B}$, the inner product can be rewritten as the *weighted product*

$$\langle u, v\rangle = 3\,\tilde{u}_1\tilde{v}_1 + 8\,\tilde{u}_2\tilde{v}_2.$$

("Tilde" on u, v indicate our use of coordinates with respect to $\mathcal{B}$.) Explain where the numbers 3 and 8 come from.

Note: Do not compute $\mathcal{B}$; although, you may indicate where $\mathcal{B}$ comes from.

In class we talked about orthogonal diagonalization of a symmetric matrix: $A = Q\Lambda Q^T$,

which allows rewriting $\langle u|v\rangle = u^TAv = u^TQ\Lambda Q^Tv$

as $\langle u|v\rangle = \tilde{u}^T\Lambda\tilde{v}$ where $\tilde{u} := Q^Tu$, $\tilde{v} := Q^Tv$

"Substitution"

Here $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ (λ_i eigenvalues)

so $\langle u|v\rangle = \lambda_1\tilde{u}_1\tilde{v}_1 + \lambda_2\tilde{u}_2\tilde{v}_2$

Hence 3, 8 are the eigenvalues of A.

Moreover, $\tilde{u}$ arises from u from $\tilde{u} = Q^Tu$ so Q^T is the change of basis matrix, $P_{\mathcal{B}\leftarrow st} = Q^T$.

Thus $P_{st\leftarrow\mathcal{B}} = (Q^T)^{-1} = Q$, which means that $\mathcal{B}$ is made of columns of Q.

since Q orthogonal!

3.[10pts] Consider the space of polynomials with the funky inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\underbrace{(1-x^2)}_{\text{a funky weight}}dx.$$

Jacobi polynomials arise from applying the Gram-Schmidt process to $\{1, x, x^2, \ldots\}$. The first two are $J_0(x) = 1$, $J_1(x) = x$. Find the third, $J_2(x)$.

Note: Mind that J_0 and J_1 are not normalized! Also, save time by using that the integral is zero if the integrand is an odd function.

I went overboard here: Only 1, 4, 5 will be used.

(1) $\|1\|^2 = \int_{-1}^{1} 1\cdot 1\,(1-x^2)dx = x - \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{4}{3}$

(2) $\|x\|^2 = \int_{-1}^{1} x\cdot x\cdot(1-x^2)dx = \frac{1}{3}x^3 - \frac{1}{5}x^5\Big|_{-1}^{1} = 2\left(\frac{1}{3}-\frac{1}{5}\right) = \frac{4}{15}$

(3) $\langle 1|x\rangle = \int_{-1}^{1} 1\cdot x\,(1-x^2)dx = 0$ because the integrand is odd.

(4) $\langle 1|x^2\rangle = \int_{-1}^{1} 1\cdot x^2(1-x^2)dx = \boxed{\text{Done above}} = \frac{4}{15}$

(5) $\langle x|x^2\rangle = \int_{-1}^{1} x\cdot x^2(1-x^2)dx = 0$ because integrand is odd.

Solution starts here:

J_2 is found from: $J_2 = \text{normalized}\left(J_2 - \mathrm{pr}_{\mathrm{span}(1,x)} J_2\right)$.

So $J_2 = \text{norm.}\left(x^2 - \frac{\langle 1|x^2\rangle}{\|1\|^2}\cdot 1 - \frac{\langle x|x^2\rangle}{\|x\|^2}\cdot x\right)$

$= \text{norm.}\left(x^2 - \frac{4}{15}\cdot\frac{3}{4}\cdot 1 - 0\right)$

$= \text{normalized}\left(x^2 - \frac{1}{5}\right)$ where I used only (1), (4), (5)

Since I did not insist on normalization $\boxed{\text{Answer: } x^2 - \frac{1}{5}}$

(If you want normalized J_2, we need $\|x^2 - \frac{1}{5}\| = \langle x^2 - \frac{1}{5} | x^2 - \frac{1}{5}\rangle$

$= \langle x^2|x^2\rangle - 2\langle \frac{1}{5}|x^2\rangle + \langle \frac{1}{5}|\frac{1}{5}\rangle = \ldots$)

Also make sure you know the condition number discussion on p 585 in the book

4.[10pts] Suppose that the right hand side in $Ax = b$ is perturbed to $b' = b + \Delta b$ so that the new solution $x' = x + \Delta x$ satisfies $Ax' = b'$. Assuming A is invertible, show (justifying all steps) that

$$(\star) \qquad \frac{\|\Delta x\|}{\|x\|} \leq \operatorname{cond}(A)\frac{\|\Delta b\|}{\|b\|}. \qquad \text{Recall } \operatorname{Cond}(A) := \|A\| \cdot \|A^{-1}\|$$

We have

$$A(x + \Delta x) = b + \Delta b,$$
$$Ax + A\Delta x = b + \Delta b.$$

Subtracting $Ax = b$ yields

$$A\Delta x = \Delta b \text{ so } \Delta x = A^{-1}\Delta b$$

property of "compatibility"

$$\text{so } \|\Delta x\| = \|A^{-1}\Delta b\| \leq \|A^{-1}\| \cdot \|\Delta b\|. \qquad (*)$$

At the same time $b = Ax$ gives $\|b\| = \|Ax\| \leq \|A\| \cdot \|x\| \qquad (**)$

Multiplying $(*)$ and $(**)$ side by side we get

$$\|\Delta x\| \cdot \|b\| \leq \|A^{-1}\| \cdot \|A\| \cdot \|\Delta b\| \cdot \|x\| \quad \text{which is } (\star).$$

5.[15pts] For $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. compute the following matrix norms

$\|A\|_1 = \ldots$ max col. sum $= \max\{1+0, 1+2\} = 3$

$\|A\|_\infty = \ldots$ max row sum $= \max\{1+1, 0+2\} = 2$

$\|A\|_F = \ldots \sqrt{1^2 + 1^2 + 0^2 + 2^2} = \sqrt{6}$

$\|A\|_2 = \ldots \sigma_1 = \sqrt{\frac{3+\sqrt{5}}{2}}$ See BELOW

Note: This last one is the operator norm with respect to the usual Euclidean length and requires more substantial work.

Computation of σ_1 (the first singular value)

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} \qquad \begin{matrix} \text{Trace} = 6 \\ \text{Det} = 4 \end{matrix} \qquad \text{so } \lambda^2 - 6\lambda + 4 = 0$$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{36-16}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{\frac{3+\sqrt{5}}{2}}$$

6.[10pts] Let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ and A be the 4×1 matrix of all 1's, $A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Find the **least squares solution** to the system $Ax = b$.

Note: x is 1×1, so it is just a scalar! Still, the general theory applies.

$$A^T A = [4] \quad \text{and} \quad A^T b = b_1 + b_2 + b_3 + b_4$$

so normal equations $A^T A \bar{x} = A^T b$ become

$$4\bar{x} = b_1 + \dots + b_4$$

$$\boxed{\text{Answer: } \bar{x} = \frac{b_1 + b_2 + b_3 + b_4}{4}}$$

the average!

NOTE: We "just solved" the system $\begin{cases} x = b_1 \\ x = b_2 \\ x = b_3 \\ x = b_4 \end{cases}$

7.[10pts] Assuming $A = QR$ is the QR-decomposition of A (with linearly independent columns), carefully derive the expression for the the **least squares solution** $\bar{x}$ to the system $Ax = b$ in terms of Q and R and b?

Note: You can use the "normal equations" as the departure point of your derivation.

$A^T A \bar{x} = A^T b$ become

$$(QR)^T QR \bar{x} = (QR)^T b$$

$$R^T \underbrace{Q^T Q}_{I} R \bar{x} = R^T Q^T b$$

$$R^T R \bar{x} = R^T Q^T b \qquad / (R^T)^{-1}!$$

$$R\bar{x} = Q^T b \qquad / R^{-1}.$$

$$\bar{x} = R^{-1} Q^T b$$

where we used that (due to columns of A being independent) R is invertible and thus R^T is invertible.

(Recall $(R^T)^{-1} = (R^{-1})^T$ if R^{-1} exists.)

8.[20pts] Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

a)[8pts] Compute (smartly) the singular values of A.

A^TA is 4×4, ... too big.

So we look at $AA^T = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ Trace $= 5$, Det $= 5$

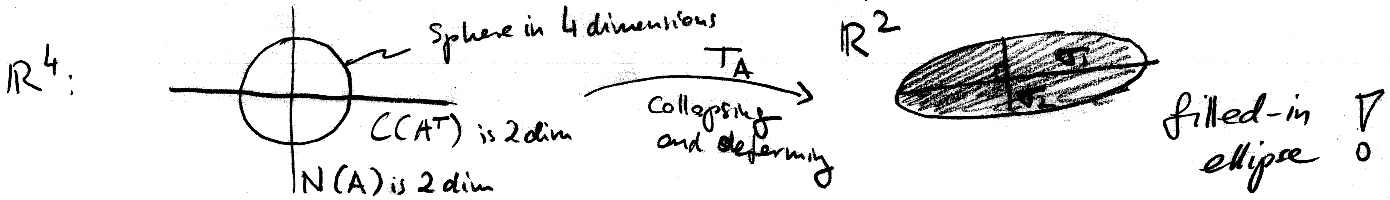
$\lambda^2 - 5\lambda + 5 = 0$

$\lambda_{1,2} = \frac{5 \pm \sqrt{25-20}}{2}$

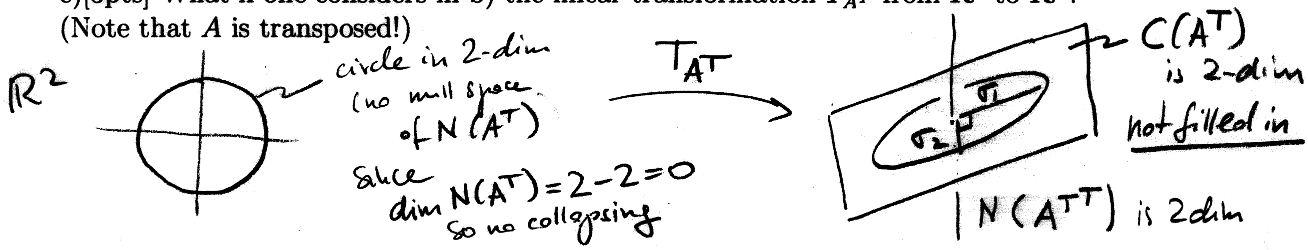
$\sigma_1 = \sqrt{\frac{5+\sqrt{5}}{2}}$; $\sigma_2 = \sqrt{\frac{5-\sqrt{5}}{2}}$

b)[4pts] What geometric figure is the image of the unit sphere (in $\mathbf{R}^4$) under the linear transformation T_A from $\mathbf{R}^4$ to $\mathbf{R}^2$?

~~Note: OK, you know it is an "ellipse" or "filled-in ellipse". Tell me which of the~~ two and why (based on a suitable theorem, if you wish).



c)[3pts] What if one considers in b) the linear transformation T_{A^T} from $\mathbf{R}^2$ to $\mathbf{R}^4$? (Note that A is transposed!)



d)[5pts] Find the direction vector (or slope) of the long axis of the ellipse from b). (Hint: Use that the u_i are eigenvectors of AA^T.)

From a): $AA^T = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ lets find the eigenvector $\vec{u}_1 = \begin{bmatrix} x \\ y \end{bmatrix}$

from $(AA^T - \lambda_1 I)\vec{u}_1 = 0$ which is $\begin{cases} (3-\lambda_1)x + y = 0 \\ 1 \cdot x + (2-\lambda_1)y = 0 \end{cases}$

These two equations are linearly dependent so we drop one (say the second one) and are left with:

$y = -(3-\lambda_1)x$

so the slope $= -3 + \lambda_1 = -3 + \frac{5+\sqrt{5}}{2} = \frac{\sqrt{5}-1}{2}$.