SAMPLE (deviates a bit from the one administered)

Math 333 Second Exam (20 Nov 2012)

Name:

Show all work (unless instructed otherwise). Good Luck!

0.[8pts] Circle **True** or **False** without explanation:

(**T** or **F**) For the identity matrix I and <u>any matrix norm</u>, $\|I\| = 1$.

(**T** or **F**) A has rank 10 iff A has (exactly) 10 non-zero singular values.

(**T** or **F**) The pseudo-inverse A^+ is an invertible matrix.

(**T** or **F**) For any u, v in an inner product space, we have $\langle u|v\rangle \le \|u\|\|v\|$.

1.[15pts] Suppose that u, v are vectors in an inner product space such that

$$\|u\| \overset{(*)}{=} \|v\| \quad \text{and} \quad \langle u|v\rangle = -3.$$

Use the properties of the inner product (not just specific examples) to solve:

a) Evaluate $\|u+v\|^2 - \|u-v\|^2 = \ldots \langle u+v|u+v\rangle - \langle u-v|u-v\rangle$

"foil"

$= \langle u|u\rangle + 2\langle u|v\rangle + \langle v|v\rangle - \langle u|u\rangle + 2\langle u|v\rangle - \langle v|v\rangle$

$= 4\langle u|v\rangle = 4\cdot(-3) = -12$

b) Show that $u+v$ and $u-v$ are perpendicular.

$\langle u+v|u-v\rangle \underset{\text{"foil"}}{=} \langle u|u\rangle + \cancel{\langle v|u\rangle - \langle u|v\rangle} - \langle v|v\rangle$

by symmetry

$= \|u\|^2 - \|v\|^2 = 0$ by $(*)$

2.[20pts] For $u, v \in \mathbf{R}^2$, define $\langle u, v\rangle := 4u_1v_1 + u_1v_2 + u_2v_1 + 4u_2v_2$.

a) Verify that $\langle u, u\rangle \geq 0$ for all $u \in \mathbf{R}^2$.

$$\langle u | u \rangle = 4u_1^2 + 2u_1u_2 + 4u_2^2 \underset{\text{Complete the square}}{=} \underbrace{4\left(u_1 + \tfrac{1}{4}u_2\right)^2}_{\geq 0} + \underbrace{\left(4 - \tfrac{1}{4}\right)u_2^2}_{\geq 0} \geq 0$$

as perfect squares so

b) Give the matrix A with $\langle u, v\rangle = u^T A v$.

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

c) Find the lengths of the semi-axes for the unit circle (for this inner product).

Need eigenvalues of A: Trace $= 8$, Det $= 4 \cdot 4 - 1 \cdot 1 = 15$

$\lambda^2 - 8\lambda + 15 = 0$

$(\lambda - 3)(\lambda - 5) = 0$

$\lambda_1 = 5,\ \lambda_2 = 3$

This means that $u^T A u = 1$ can be rewritten (in new coordinates) as

$5\tilde{u}_1^2 + 3\tilde{u}_2^2 = 1$ which is

$\left(\dfrac{\tilde{u}_1}{\frac{1}{\sqrt{5}}}\right)^2 + \left(\dfrac{\tilde{u}_2}{\frac{1}{\sqrt{3}}}\right)^2 = 1$ So the semi-axes are $\frac{1}{\sqrt{5}}$ and $\frac{1}{\sqrt{3}}$

d) Find the slopes of the semi-axes for the unit circle (for this inner product).

The semi-axes point in the direction of the eigenvectors of A.

The eigenequation $(A - \lambda I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is

$$\begin{cases} (4-\lambda)x + y = 0 \\ x + (4-\lambda)y = 0 \end{cases}$$

where the equations are (always) redundant so we take one:

$$y = \underbrace{(\lambda - 4)}_{\text{slope}} x$$

The two slopes are $\lambda_1 - 4 = 5 - 4 = 1$

and $\lambda_2 - 4 = 3 - 4 = -1$

3.[15pts] Consider the space of polynomials with the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\frac{dx}{\sqrt{1-x^2}}.$$

Chebyshev polynomials arise from the Gram-Schmidt process applied to $\{1, x, x^2, \ldots\}$. The first three, unnormalized, are

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1.$$

Using the following: $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi, \quad \int_{-1}^{1} \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{1}{2}\pi, \quad \int_{-1}^{1} \frac{x^4 dx}{\sqrt{1-x^2}} = \frac{3}{8}\pi,$

a) find the norms

$\|1\| = \ldots \left(\int_{-1}^{1} 1 \cdot 1 \frac{dx}{\sqrt{1-x^2}}\right)^{1/2} = \pi^{1/2} = \sqrt{\pi}$

$\|x\| = \ldots \left(\int_{-1}^{1} x \cdot x \frac{dx}{\sqrt{1-x^2}}\right)^{1/2} = \sqrt{\frac{1}{2}\pi}$

b) without computing any anti-derivatives, justify $\langle x|1\rangle = 0$ and $\langle x|x^2\rangle = 0$;

$\langle x|1\rangle = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} dx$

$\langle x|x^2\rangle = \int_{-1}^{1} \frac{x \cdot x^2}{\sqrt{1-x^2}} dx$

these both have integrands that are odd functions (i.e. $f(-x) = -f(x)$). Because the integration is over the segment $[-1,1]$ that is symmetric about 0, it must be zero.

important $\rightarrow$ $\operatorname{span}(T_0, T_1, T_2)$ $=$

c) find the projection of x^3 onto span$\{1, x, x^2\}$. (Indicate the formula used.)

$$p = \frac{\langle x^3|1\rangle}{\langle 1|1\rangle} \cdot 1 + \frac{\langle x^3|x\rangle}{\langle x|x\rangle} x + \frac{\langle x^3|2x^2-1\rangle}{\langle 2x^2-1|2x^2-1\rangle}(2x^2-1)$$

Where $\langle x^3|1\rangle = 0$ and $\langle x^3|2x^2-1\rangle = 0$ for the same reason as in b) (i.e. the integrands are odd).

Hence $$p = \frac{\langle x^3|x\rangle}{\langle x|x\rangle} \cdot x = \frac{\int_{-1}^{1} x^4 \frac{dx}{\sqrt{1-x^2}}}{\int_{-1}^{1} x^2 \frac{dx}{\sqrt{1-x^2}}} \cdot x = \frac{\frac{3}{8}\pi}{\frac{1}{2}\pi} \cdot x = \frac{3}{4}x.$$

4.[10pts] Carefully prove that $N(A^TA) = N(A)$ for any rectangular matrix A.

To see $N(A) \subseteq N(A^TA)$ it suffices to observe that if $Ax = 0$ then $A^TAx = A^T0 = 0$.

To see $N(A^TA) \subseteq N(A)$ we have to show that $A^TAx = 0$ $(*)$ implies $Ax = 0$ (and we cannot just multiply by $(A^T)^{-1}$ for it may fail to exist!).

From $(*)$ we get $x^TA^TAx = x^T0 = 0$, which is

$(Ax)^TAx = 0$ or, equivalently,

$\|Ax\|^2 = 0$.

Hence $\|Ax\| = 0$, and so $Ax = 0$, as desired.

5.[15pts] For $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, compute the following matrix norms

$\|A\|_1 = \ldots$ "max col. sum" $= \max\{2+4, 3+1\} = 6$

$\|A\|_\infty = \ldots$ "max row sum" $= \max\{2+3, 4+1\} = 5$

$\|A\|_F = \ldots \sqrt{2^2+3^2+4^2+1^2} = \sqrt{4+9+16+1}$

6.[10pts] Certain $n \times n$ matrix A has $\lambda_1 = 5$ and $\lambda_n = 1/4$ as its eigenvalues.

a) What can you say about the operator norm $\|A\| := \max_{\|x\|=1} \|Ax\|$? Explain.

$$\|A\| \geq |\lambda_1| = 5$$

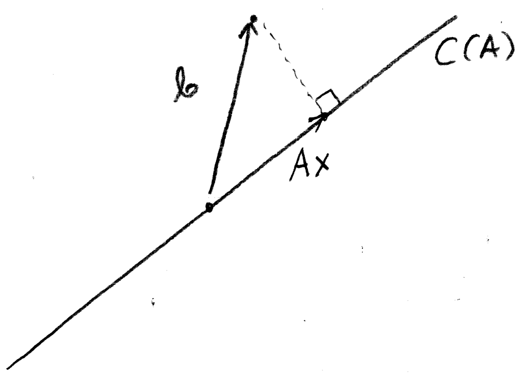
This is because if x_0 is a normalized eigenvector of λ_1, i.e. $Ax_0 = \lambda_1 x_0$, then $\max_{\|x\|=1} \|Ax\| \geq \|Ax_0\| = \|\lambda_1 x_0\| = |\lambda_1| \underbrace{\|x_0\|}_{1} = |\lambda_1|$.

b) What can you say about the condition number cond(A) of this A? Explain.

$$\text{Cond}(A) = \|A\| \cdot \|A^{-1}\| \geq |\lambda_1| \cdot |\lambda_n^{-1}| = 5 \cdot \left(\tfrac{1}{4}\right)^{-1} = 5 \cdot 4 = 20$$

where we used a) for $\|A\| \geq |\lambda_1|$
and then a version of a) with A replaced by A^{-1} to get $\|A^{-1}\| \geq |\lambda_n^{-1}|$
(Here it is crucial that the eigenvalues of A^{-1} include λ_1^{-1} and λ_n^{-1}.)

7.[10pts] The normal equation $A^T Ax = A^T b$ is an expression of perpendicularity of a certain vector to the column space $C(A)$. What vector is this? Draw a figure and reproduce the logical steps that yield $A^T Ax = A^T b$.



From the figure, we have

$b - Ax \perp C(A)$, meaning

$\forall_{y \in \mathbb{R}^n}$ $b - Ax \perp Ay$, which amounts to vanishing of dot product

$\forall_{y \in \mathbb{R}^n}$ $(Ay)^T(b - Ax) = 0$.

Thus $\forall_{y \in \mathbb{R}^n}$ $y^T(A^T b - A^T A x) = 0$,

which is only possible if $A^T b - A^T A x = 0$, i.e.,

$$A^T A x = A^T b.$$

8.[10pts] For $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$, find the following:

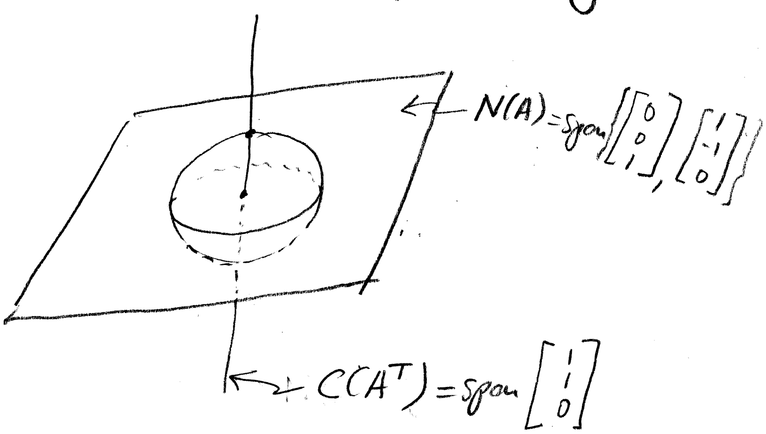
a) the operator norm $\|A\|_2 := \max_{\|x\|=1} \|Ax\|$ were $\|x\|$ is the ordinary length of x.

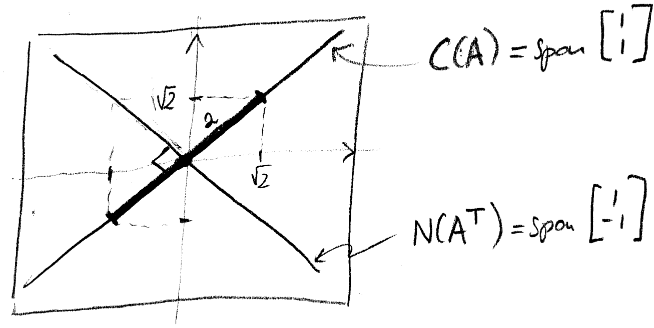
$A^TA = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ Trace $= 4$, Det $= 0$; $\lambda^2 - 4\lambda = 0$, $\lambda(\lambda - 4) = 0$ so $\lambda_1 = 4$, $\lambda_2 = 0$

$\|A\|_2 =$ the largest singular value $= \sigma_1 = \sqrt{\lambda_1} = \sqrt{4} = 2$

b) the image of the unit ~~circle~~ sphere under T_A (which sends x to Ax).

The image is a "filled-in ellipsoid" in $C(A) = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ with the largest (and only) semi-axis of length $\sigma_1 = 2$.





9.[10pts] Prove that if a <u>non-zero</u> λ is an eigenvalue of AA^T then λ is also an eigenvalue of A^TA. *(This is a part of the argument that A and A^T have the same singular values, #30 page 632.)*

Suppose λ is an eigenvalue of AA^T, i.e., $AA^Tv = \lambda v$ (1) for some $v \neq 0$ (2). We also assume $\lambda \neq 0$ (3).

By applying A^T to both sides of (1), we get

$$A^TAA^Tv = A^T\lambda v \quad \text{, which is}$$

(4) $$(A^TA)A^Tv = \lambda A^Tv.$$

Note that, $\tilde{v} := A^Tv$ is non-zero since if it were 0 then (1) would read $A \cdot 0 = \lambda v$ so $\lambda v = 0$, but $\lambda v \neq 0$ by (2) and (3).

Ok, so (4) is $A^TA\tilde{v} = \lambda\tilde{v}$ with $\tilde{v} \neq 0$,

which means exactly that <u>λ is an eigenvalue of A^TA.</u>