0	1	2	3	4	5	6	7	8	9	
10	10	10	10	10+	10	10	10	5	10	95

Math 333 First Exam (19 Oct 2010)

median score 62
(65%)
avg score 59p
(62%)

Name:

If you use a theorem indicate what it is. Show all work (unless instructed otherwise).
Good Luck!

2pts each

0. Circle **True** or **False** without explanation:

(**T** or **F**) Every linearly independent set in V can be extended to a basis of V.

(**T** or **F**) $\mathbf{R}^9$ and $M_{3\times 3}$ are isomorphic.

(**T** or **F**) If $T \circ S$ is one-to-one then so is T.

(**T** or **F**) If U, W are subspaces then so is their union $U \cup W$.

(**T** or **F**) Any linear $T : \mathcal{P}_2 \to \mathcal{P}_2$ that is one-to-one is onto.

1. Find a basis and the dimension for the subspace (in $M_{3\times 3}$) given by

$$W := \left\{ \begin{bmatrix} a & 0 & c \\ c & a & b \\ 0 & b & a \end{bmatrix} : a, b, c \in \mathbf{R} \right\}.$$ Do not prove anything.

$$\| $$

$$a\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + b\begin{bmatrix} & & \\ & & 1 \\ & 1 & \end{bmatrix} + c\begin{bmatrix} & & 1 \\ 1 & & \\ & & \end{bmatrix}$$

basis (clearly span and are lin. indep)

so dim = 3.

10pts 2. Carefully verify that the set of all even functions on $\mathbf{R}$,

$$W := \{f \in \mathcal{F} : f(x) = f(-x) \text{ for all } x \in \mathbf{R}\},$$

is a subspace of $\mathcal{F}$.

Suppose $f, g \in W$ and c is a scalar in $\mathbb{R}$.

Then $(f+g)(-x) \underset{\text{def of } +}{=} f(-x) + g(-x) \underset{\text{hyp. on } f \text{ and } g}{=} f(x) + g(x) \underset{\text{def of } +}{=} (f+g)(x)$

$(cf)(-x) = c \cdot f(-x) = c \cdot f(x) = (cf)(x)$

Thus $f+g \in W$ and $c \cdot f \in W$ making W a subspace.

10pts 3. Suppose that $T : V \to W$ is linear and $\{T(v_1), \ldots, T(v_n)\}$ is a linearly independent set. Prove that $\{v_1, \ldots, v_n\}$ is linearly independent but the opposite implication may fail.

6pts

Suppose $a_1 v_1 + \cdots + a_n v_n = 0$ for some $a_1, \ldots, a_n$.

Then $T(\text{——} \text{"} \text{——}) = T(0)$ i.e.

$a_1 T(v_1) + \cdots + a_n T(v_n) = 0$ (by linearity of T and $T(0) = 0$).

By our hypothesis on $T(v_i)$, $a_1 = a_2 = \cdots = a_n = 0$.

Thus $\{v_1, \ldots, v_n\}$ are lin. indep.

4pts

The opposite implication fails as shown by an example

$T : \mathbb{R}^2 \to \mathbb{R}^2$ being the zero map, ie., $T(v) = 0$ for all $v \in \mathbb{R}^2$.

Just take $v_1 = e_1$, $v_2 = e_2$.

5pts

4. Let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be given by $T(p) := p''$ (the second derivative).
a) Describe kernel(T) and compute nullity(T).

$\ker(T) = \{ p \in \mathcal{P}_2 : p'' = 0 \} = \{ ax^2 + bx + c : 2a = 0 \} = \{ bx + c : b, c \in \mathbb{R}$

which are all "linear functions" as encountered in Calculus class; or, simply, $\mathcal{P}_1$

nullity(T) = . 2 since $\{1, x\}$ is a basis of $\ker(T)$

5pts b) Describe range(T) and compute rank(T).

range $(T) = \{ p'' : p \in \mathcal{P}_2 \} = \{ 2a : a \in \mathbb{R} \} =$ all constant functions or, simply, $\mathcal{P}_0$.

rank(T) = . . 1 since $\{1\}$ is a basis of range (T).

c) Write out the matrix of T in the standard basis $\{1, x, x^2\}$.

Since $T(1) = 0$
$T(x) = 0$ we have $[T] = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$T(x^2) = 2 = 2 \cdot 1$

10pts 5. You learned that if $\dim V = \dim W$ (and finite) then V and W are isomorphic. Recall the construction of the isomorphism $T : V \to W$ (without proving its properties).

Let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of V and $\{w_1, \dots, w_n\}$ be a basis of W.

Then, on any $v \in W$, we define

Can cut out this piece.

$$T(v) = \boxed{T\left(\sum_{i=1}^{n} x_i v_i\right) = \sum_{i=1}^{n} x_i T(v_i)} = \sum_{i=1}^{n} x_i w_i$$

where x_i are the coordinates of v i.e. $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [v]_{\mathcal{B}}$.

4 pts

6. Suppose that $\{v_1, v_2, v_3\}$ is a basis (of some V).
a) Show that $\mathcal{B} = \{v_1 - v_2, v_2 - v_3, v_3 - v_1\}$ is not a basis.

The vectors are lin. dep. since:

$$1\cdot(V_1 - V_2) + 1\cdot(V_2 - V_3) + 1\cdot(V_3 - V_1)$$
$$= V_1 - V_2 + V_2 - V_3 + V_3 - V_1 = 0.$$

6 pts

b) Show that $\mathcal{B} = \{v_1 + v_2, v_2 + v_3, v_3 + v_1\}$ is a basis.

Lin. independence: $a_1(V_1+V_2) + a_2(V_2+V_3) + a_3(V_3+V_1) = 0$

$(a_1 + a_3)V_1 + (a_1+a_2)V_2 + (a_2+a_3)V_3 = 0$

so $\begin{cases} a_1 + a_3 = 0 \\ a_1 + a_2 = 0 \\ a_2 + a_3 = 0 \end{cases}$ i.e. $\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{\text{matrix } A \ 3\times 3} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = 0$ so $a_1 = a_2 = a_3 = 0$.

matrix A of $\det = 1 + 1 - 0 = 2$
3x3 so A^{-1} exists.

Spanning: Any v can be written as $b_1V_1 + b_2V_2 + b_3V_3$.
The question is if we can write it as $a_1(V_1+V_2) + a_2(V_2+V_3) + a_3(V_3+V_1)$.
That is if we can solve $A\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. But this we can $a = A^{-1}b$.

10 pts

7. Suppose that the set $\mathcal{A} = \{v_1, \ldots, v_n\}$ spans all of V but fails to do so if one removes any of its vectors. Prove that $\mathcal{A}$ is independent.

Suppose A is dependent i.e. $a_1V_1 + \ldots + a_nV_n = 0$ and some $a_i \neq 0$.

Then $V_i = -\frac{1}{a_i}(a_1V_1 + \ldots + a_nV_n)$ so $V_i \in \operatorname{Span}(V_1, \ldots, V_n)$
ith missing ith missing

Thus $\operatorname{span}(V_1, \ldots, V_n) = V$
ith removed

5 pts 8. Show that $\{e^x, e^{-x}\}$ is linearly independent (in $\mathcal{F}$).

Say $ae^x + be^{-x} = 0$.

Then @ $x=0$: $a + b = 0$

@ $x=1$: $a \cdot e + b \cdot e^{-1} = 0$

so $\underbrace{\begin{bmatrix} 1 & 1 \\ e & e^{-1} \end{bmatrix}}_{\text{has det} = e^{-1} - e \neq 0} \begin{bmatrix} a \\ b \end{bmatrix} = 0$ so $a = b = 0$.

9. A certain binary code sends a message $x = (x_1, x_2)^T$ to $b := Gx$ where the code generating matrix G is

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{so} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = Gx = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \\ x_1 \end{bmatrix}$$

6 pts a) Find the conditions ("check") for (b_1, b_2, b_3, b_4) to be an uncorrupted message (i.e. $b = Gx$ for some x).

b uncorrupted iff $b \in C(G)$ iff $b \perp N(G^T)$

$N(G^T)$ is 2-dim (since G has two pivots $2 = 4 - 2$) and we "easily guess"

$N(G^T) = \text{span}\{[1,1,1,0], [1,0,0,1]\}$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}}_{\substack{\parallel \\ [A|I]}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = 0 \quad \text{i.e.} \quad \begin{array}{l} b_1 + b_2 + b_3 = 0 \leftarrow \text{ordinary parity check} \\ b_1 + b_4 = 0 \leftarrow b_1 = b_4 \end{array}$$

NOTE: $[A|I]$ per theorem we proved so could short circuit

4 pts. b) How many bits in (b_1, b_2, b_3, b_4) can be corrupted and the original message (x_1, x_2) is still recoverable? (Give an answer and some indication of the reason.)

Just one since one can recover x_1, x_2 from any three (in fact any two of the first three) and corrupting the middle bits can be unrecoverable e.g.

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 \\ x \\ x \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} 1 \\ x \\ x \\ 1 \end{bmatrix}$$

0	1	2	3	4	5	6	7	8	
8	10	10	10	10	10	10	10	10	88 max

Math 333 Second Exam (30 Nov 2010)

Name:

Show all work (unless instructed otherwise). Good Luck!

0. Circle **True** or **False** without explanation: Below vectors are in a finite dimensional vector space V, $\langle\cdot|\cdot\rangle$ denotes an inner product, and $\|\cdot\|$ its associated norm.

(**T** or **F**) $\langle u|v\rangle = \|u\|\|v\|$ only if one of u and v is a scalar multiple of another.

(**T** or **F**) Every norm on V comes from an inner product.

(**T** or **F**) $\overline{x} := A^+b$ is a **least squares solution** to $Ax = b$.

(**T** or **F**) For any square matrix A, the SVD gives a diagonalization of A.

10pts 1. Suppose that u, v, w are vectors in an inner product space and

$$\langle u|v\rangle = -3, \quad \langle u|w\rangle = -6, \quad \langle v|w\rangle = 6$$

$$\|u\| = 3, \quad \|v\| = \sqrt{5}, \quad \|w\| = \sqrt{8}.$$

5pts a) Evaluate $\|u + w\| = \ldots \sqrt{\langle u+w \mid u+w\rangle}$

$$= \left(\langle u|u\rangle + 2\langle u|w\rangle + \langle w|w\rangle\right)^{1/2}$$

$$= \left(3 + 2\cdot(-3) + 8\right)^{1/2} = \boxed{\sqrt{5}}$$

5pts b) Determine by computation if $u + v$ is perpendicular to w.

$$\langle u+v|w\rangle = \langle u|w\rangle + \langle v|w\rangle$$

$$= -6 + 6 = 0 \quad \text{so } \boxed{\text{yes}}$$

10 pts 4. Let $\|A\|$ be a matrix norm of a square matrix A compatible with a vector norm $\|x\|$. Show that if λ is an eigenvalue of A then

$$\|A\| \geq |\lambda|.$$

Let $v \neq 0$ be the corresponding eigenvector so that

$$Av = \lambda v.$$

Then $\|A\| \cdot \|v\| \underset{\text{by compatibility}}{\geq} \|Av\| = \|\lambda v\| \underset{\text{prop. of } \|\cdot\|}{=} |\lambda| \cdot \|v\|$

Div. by $\|v\|$, yields $\|A\| \geq |\lambda|$.

10 pts 5. Prove (while justifying each step) that the Frobenius norm

$$\|A\|_F := \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}$$

is compatible with the Euclidean norm, i.e., $\|Ax\| \leq \|A\|_F \|x\|$ for all x in $\mathbf{R}^n$.

$$\|Ax\|^2 = \sum_i \Big| \sum_j a_{ij} x_j \Big|^2$$

$$\leq \sum_i \left(\sqrt{\sum_j a_{ij}^2} \sqrt{\sum_j x_j^2} \right)^2$$

by using Cauchy Schwarz

$$= \sum_i \left(\sum_j a_{ij}^2 \sum_j x_j^2 \right)$$

$$= \left(\sum_i \sum_j a_{ij}^2 \right) \sum_j x_j^2$$

$$= \sum_{i,j} a_{ij}^2 \sum_i x_i^2 = \|A\|_F^2 \cdot \|x\|^2$$

6 pts

8. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. lengths only!

a) Compute the semi-axis of the ellipse that is the image of the unit circle under the linear transformation T_A of the plane $\mathbf{R}^2$ induced by A.

$$A^T A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$ has char. poly

$$\lambda^2 - \text{trace}\,\lambda + \det = 0$$

$$\lambda^2 - 3\cdot\lambda + 1 = 0$$ with roots $\lambda_{1,2} = \dfrac{3 \pm \sqrt{9-4\cdot 1}}{2}$

Singular values : $\sigma_1 = \sqrt{\lambda_1} = \sqrt{\dfrac{3+\sqrt{5}}{2}}$

$\sigma_2 = \sqrt{\lambda_2} = \sqrt{\dfrac{3-\sqrt{5}}{2}}$ are the semi-axis lengths

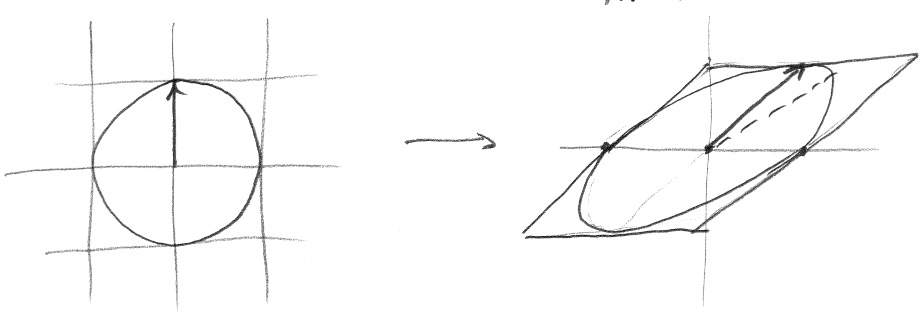
3 pts

b) What is the value of the operator norm $\|A\| := \max_{\|x\|=1} \|Ax\|$? (Here $\|x\|$ is the ordinary Euclidean norm.)

$$\|A\| = \sigma_1 = \sqrt{\frac{3+\sqrt{5}}{2}}.$$

1 pt

c)* Sketch the image of the square $\{(x, y) : -1 \le x \le 1, -1 \le y \le 1\}$ under T_A and guess-sketch the ellipse (from a)). mark the semi-axes



12:35

0	1	2	3	4	5	6	7	8	9	10	
10	10	10	10	10	10	15	10	15	10	10	120 max

Math 333 Final Exam (15 Dec 2010)

Name:

Show all work (unless instructed otherwise). Good Luck!

10pts

0. Circle **True** or **False** without explanation. Below V is a finite dimensional vector space.

(**T** or **F**) V is isomorphic to $\mathbf{R}^n$ provided $n = \dim(V)$.

(**T** or **F**) Any two bases of V have the same number of elements.

(**T** or **F**) If $T : V \to V$ is linear and onto then it is one-to-one.

(**T** or **F**) For orthogonally diagonalizable A, the singular values are eigenvalues.

(**T** or **F**) For $A \geq 0$, $A^3 > 0$ implies that A is irreducible.

10pts

1. V is an inner product space and $\|x\| := \sqrt{\langle x|x\rangle}$ is the associated norm.
a) Demonstrate the Parallelogram Law:

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2 \quad \text{for all } u, v \text{ in } V.$$

$$\text{LHS} = \langle u+v|u+v\rangle^2 + \langle u-v|u-v\rangle^2$$

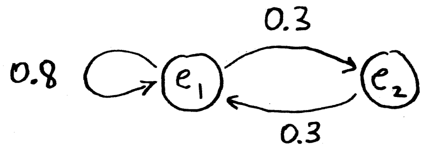
$$= \|u\|^2 + \cancel{2\langle u|v\rangle} + \|v\|^2 + \|u\|^2 - \cancel{2\langle u|v\rangle} + \|v\|^2$$

$$= 2\|u\|^2 + 2\|v\|^2$$

adjusted E1 score = max$\{$E1, 0.2E1+0.8M$)$

makeup

10pts 2. The following graph describes the flow of wealth in a partnership of two agents e_1 and e_2.



$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.3 & 0 \end{bmatrix} \qquad \text{trace} = 0.8 \quad \det = -0.09$$

a) What is the asymptotic ratio of the wealth of e_1 to the wealth of e_2?

$$\lambda^2 - 0.8\lambda - 0.09 = 0 \quad \text{so} \quad \lambda = \frac{0.8 \pm \sqrt{0.64 + 0.36}}{2} = \frac{0.8 \pm 1}{2}$$

so $\lambda_1 = 0.9$; $\lambda_2 = -0.1$

$$A - \lambda_1 I = \begin{bmatrix} -0.1 & 0.3 \\ 0.3 & 0.9 \end{bmatrix} \quad \text{so} \quad -0.1x_1 + 0.3x_2 = 0$$

gives eigenvector $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ hence $\boxed{\text{the ratio} = \frac{3}{1}}$

b) Compute the growth rate λ_1. Is this partnership going to prosper or fizzle out?

$\lambda_1 = 0.9$

Since $\lambda_1^n = 0.9^n \searrow 0$ as $n \to \infty$ the partnership fizzles out

10pts 3. Let $\|A\|$ be a matrix norm of a square matrix A compatible with a vector norm $\|x\|$. Suppose there is x with $\|x\| = 1$ such that

$$\|Ax\| = 2 \quad \text{and} \quad \|A^2x\| = 9.$$

a) Prove that $\|A\| \geq 2$.

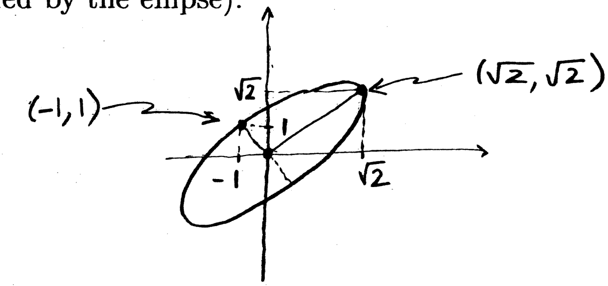
$$\|A\| = \|A\| \cdot \|x\| \overset{\text{compatibility}}{\geq} \|Ax\| = 2$$

b)* Prove that $\|A\| \geq 3$.

$$\|A\|^2 \overset{\text{matrix norm}}{\geq} \|A \cdot A\| = \|A^2\| \|x\| \geq \|A^2 x\| = 9$$

so $\|A\| \geq 3$.

10pts 4. For a 2×3 matrix A, the image of the unit sphere under T_A is the depicted solid ellipse (i.e., the oval area bounded by the ellipse).



3pts b) What is the rank(A)?

2

3pts a) What are the singular values of A? lengths of semi axes: $\sqrt{\sqrt{2}^2+\sqrt{2}^2}=\sqrt{4}=2$

and $\sqrt{(-1)^2+1^2}=\sqrt{2}$

4pts c) What is the U is in the SVD $A = U\Sigma V^T$?

$u_1 =$ normalized $(\sqrt{2},\sqrt{2})^T = (\sqrt{2}/2, \sqrt{2}/2)^T$

$u_2 =$ normalized $(-1,1)^T = (-1/\sqrt{2}, 1/\sqrt{2})^T$

10pts 5. Suppose that B is a fixed 2×2 matrix.

5pts a) Check that the set $W_B := \{A \in M_{2\times 2} : AB = BA\}$ is a subspace (of $M_{2\times 2}$).

Suppose $A, \tilde{A} \in W_B$ i.e. $AB \overset{(1)}{=} BA$ and $\tilde{A} \overset{(2)}{=} B\tilde{A}$

Then $(A+\tilde{A})B = AB + \tilde{A}B \overset{(1)(2)}{=} BA + B\tilde{A} = B(A+\tilde{A})$ so $A+\tilde{A} \in W_B$

Suppose $A \in W_B$ and α scalar.

Then $(\alpha A)B = \alpha(AB) \overset{(1)}{=} \alpha(BA) = B(\alpha A)$ so $\alpha A \in W_B$

5pts b) Find a basis of W_B for the specific B given by $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W_B$ iff $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}$

i.e. $\begin{cases} a+b = a+c & \text{so } b=c \\ a+b = b+d & \text{so } a=d \\ c+d = a+c & \text{so } d=a \\ c+d = b+d & \text{so } c=d \end{cases}$

Got $a=d$ $c=b$ i.e.

$A = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$ forms a basis

15pts 6. Let A be $n \times m$ with independent columns and QR-decomposition $A = QR$. (Here Q is $n \times m$ and R is $m \times m$.)

4pts a) Give an idea why $C(A) = C(Q)$.

From $A = QR$ columns of A are lin. comb. of col of Q, so that $C(A) \subseteq C(Q)$

Likewise, from $Q = AR^{-1}$, $C(Q) \subseteq C(A)$.

The two inclusion give $C(A) = C(Q)$

5pts b) Write the formula for the projection of $b \in \mathbf{R}^n$ onto $C(A)$ in terms of Q and b.

$$\text{proj}_{C(Q)} b = q_1^T b \cdot q_1 + \dots + q_m^T b \, q_m = QQ^T b$$

6pts c) Express the least squares solution to $Ax = b$ by using Q, R, and b. Justify.

$A^T A x = A^T b$ becomes $R^T \underbrace{Q^T Q}_{I} R x = R^T Q^T b$

$$R^T R x = R^T Q^T b \qquad / R^{-1}(R^T)^{-1}$$

$$x = R^{-1}(R^T)^{-1} R^T Q^T b = R^{-1} Q^T b$$

10pts 7. ~~Assuming $\{v_1, v_2, v_3\}$ is linearly independent, show that $\{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$~~ is linearly independent as well.

Suppose $a_1 v_1 + a_2 (v_1 + v_2) + a_3 (v_1 + v_2 + v_3) = 0$

Then $(a_1 + a_2 + a_3) v_1 + (a_2 + a_3) v_2 + a_3 v_3 = 0$

By lin indep of $\{v_1, v_2, v_3\}$: $a_3 = 0$

$a_2 + a_3 = 0$ so $a_2 = 0$

$a_1 + a_2 + a_3 = 0$ so $a_1 = 0$

15 pts 8. Consider $T : \mathcal{P}_2 \to \mathcal{P}_2$ given by $T(p(x)) = xp'(x)$.

4 pts a) What polynomials $p(x)$ make up the kernel(T)?

$x \cdot p'(x) = 0$ for all x

$p'(x) = 0$ — " —

$p(x) =$ Const polynomial.

5 pts b) Find the matrix $[T]$ of T with respect to the standard basis $\{1, x, x^2\}$.

$T(1) = x \cdot 0 = 0$

$T(x) = x \cdot 1 = x$

$T(x^2) = x \cdot 2x = 2x^2$

so
$$[T] = \begin{array}{c|ccc} & 1 & x & x^2 \\ \hline 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 \\ x^2 & 0 & 0 & 2 \end{array}$$

2 pts c) What is the rank(T)? (Use a).)

rank (T) = rank $([T]) = 2$

4 pts d) Show that $S = I + T$ is invertible. (Here I is the identity so S is explicitly given by $S(p(x)) = p(x) + xp'(x)$; although, you may just use $[S]$ in your solution.)

$$[S] = [I+T] = [I] + [T] = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$$

which is invertible and thus so is S.

10 pts 9. Construct an orthogonal basis of $\mathcal{P}_2$ with the inner product $\langle p|q\rangle := \int_0^1 p(x)q(x)dx$.

We shall orthogonalize $\{1, x, x^2\}$. Take $v_1 = 1$.

$\langle 1|x\rangle = \int_0^1 x\,dx = \frac{1}{2}x^2\Big|_0^1 = 1/2$;

Thus $\text{proj}_{\text{lin}(1)} x = \frac{\langle x|1\rangle}{\langle 1|1\rangle}\cdot 1 = \frac{1/2}{1}\cdot 1 = 1/2$ and $v_2 = x - 1/2$ is $\perp$ to v_1.

$\langle 1|x^2\rangle = \int_0^1 x^2 dx = \frac{1}{3}x^3\Big|_0^1 = 1/3$; $\langle x - 1/2|x^2\rangle = \int_0^1 x^3 - \frac{1}{2}x^2 dx$
$= \frac{1}{4}x^4 - \frac{1}{6}x^3\Big|_0^1 = \frac{1}{4} - \frac{1}{6} =$

Thus $\text{proj}_{\text{lin}(v_1,v_2)} x^2 = \frac{\langle x^2|1\rangle}{\langle 1|1\rangle}\cdot 1 + \frac{\langle x^2|x-1/2\rangle}{\langle x-1/2|x-1/2\rangle}(x - 1/2) = \frac{1}{3} + \frac{1/12}{1/2}(x - 1/2)$

$$\boxed{\text{Need } \langle x - 1/2, x - 1/2\rangle = \int_0^1 x^2 - x + 1/4\,dx = \frac{1}{3} - 1/2 + 1/4 = 1/4 - 1/6 = 1/12}$$

Hence $v_3 = x^2 - x + \frac{1}{6}$

Got $\{1, x - 1/2, x^2 - x + 1/6\}$

10 pts

7.5 pts

10. In Crystallographic Restriction Theorem, a rotation matrix $R_\theta = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ preserved some lattice Γ in $\mathbf{R}^2$, $R_\theta\Gamma = \Gamma$. Explain why trace$(R_\theta) = 2\cos\theta$ had to be an integer.

Upon choosing a basis $\{u, v\}$ with $u, v \in \Gamma$ we notice that
$R_\theta(u) = au + bv$, $R_\theta(v) = cu + dv$ for some $a, b, c, d \in \mathbb{Z}$

(This is because $R_\theta(u) \in \Gamma$ and $\Gamma = \{au + bv : a, b \in \mathbb{Z}\}$; likewise for $R_\theta(v)$.)

Thus $[R_\theta] = \begin{bmatrix} a & c \\ b & d\end{bmatrix}$ so trace R_θ = trace $[R_\theta] = a + d \in \mathbb{Z}$

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