0	1	2	3	4	5	6	7	8	
8	10	10	10	10	10	10	10	10	88 max

Math 333 Second Exam (30 Nov 2010)

Name:

Show all work (unless instructed otherwise). Good Luck!

0. Circle **True** or **False** without explanation: Below vectors are in a finite dimensional vector space V, $\langle\cdot|\cdot\rangle$ denotes an inner product, and $\|\cdot\|$ its associated norm.

(**T** or **F**) $\langle u|v\rangle = \|u\|\|v\|$ only if one of u and v is a scalar multiple of another.

(**T** or **F**) Every norm on V comes from an inner product.

(**T** or **F**) $\overline{x} := A^{+}b$ is a **least squares solution** to $Ax = b$.

(**T** or **F**) For any square matrix A, the SVD gives a diagonalization of A.

10pts

1. Suppose that u, v, w are vectors in an inner product space and

$$\langle u|v\rangle = -3, \quad \langle u|w\rangle = -6, \quad \langle v|w\rangle = 6$$

$$\|u\| = 3, \quad \|v\| = \sqrt{5}, \quad \|w\| = \sqrt{8}.$$

5pts a) Evaluate $\|u+w\| = \ldots \sqrt{\langle u+w \mid u+w\rangle}$

$$= (\langle u|u\rangle + 2\langle u|w\rangle + \langle w|w\rangle)^{1/2}$$

$$= (3 + 2\cdot(-3) + 8)^{1/2} = \boxed{\sqrt{5}}$$

5pts b) Determine by computation if $u+v$ is perpendicular to w.

$$\langle u+v \mid w\rangle = \langle u|w\rangle + \langle v|w\rangle$$

$$= -6 + 6 = 0 \qquad \text{so } \boxed{\text{yes}}$$

$\boxed{10\,\text{pts}}$ 2. Derive the **Triangle Inequality** from the **Cauchy-Schwarz Inequality**. (<u>State both</u>!)

$$\|u+v\|^2 = \langle u+v|u+v\rangle = \|u\|^2 + 2\langle u|v\rangle + \|v\|^2$$

$$\leq \|u\|^2 + 2\|u\|\cdot\|v\| + \|v\|^2 = (\|u\|+\|v\|)^2$$

$\uparrow$

(used C-S: $\langle u|v\rangle \leq \|u\|\cdot\|v\|$)

Thus $\|u+v\| \leq \|u\| + \|v\|$, which is the triangle inequality.

$\boxed{10\,\text{pts}}$ 3. Show that $\langle p(x)|q(x)\rangle := p(0)q(0) + p(1)q(1)$ is not an inner product on $\mathcal{P}_2$.

Consider $p(x) = q(x) = x(1-x)$

Then $\langle p(x)|p(x)\rangle = 0\cdot 0 + 0\cdot 0 = 0$

even though $p(x) \not\equiv 0$.

(This violates the axiom $\langle v|v\rangle = 0 \Rightarrow v = 0$.)

10 pts

4. Let $\|A\|$ be a matrix norm of a square matrix A compatible with a vector norm $\|x\|$. Show that if λ is an eigenvalue of A then

$$\|A\| \geq |\lambda|.$$

Let $v \neq 0$ be the corresponding eigenvector so that

$$Av = \lambda v.$$

Then $$\|A\| \cdot \|v\| \underset{\substack{\uparrow \\ \text{by compatibility}}}{\geq} \|Av\| = \|\lambda v\| \underset{\substack{\uparrow \\ \text{prop. of } \|\cdot\|}}{=} |\lambda| \cdot \|v\|$$

Div. by $\|v\|$, yields $\|A\| \geq |\lambda|$.

10 pts

5. Prove (while justifying each step) that the Frobenius norm

$$\|A\|_F := \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}$$

is compatible with the Euclidean norm, i.e., $\|Ax\| \leq \|A\|_F \|x\|$ for all x in $\mathbf{R}^n$.

$$\|Ax\|^2 = \sum_i \Big| \sum_j a_{ij} x_j \Big|^2$$

$$\leq \sum_i \left(\sqrt{\sum_j a_{ij}^2} \sqrt{\sum_j x_j^2} \right)^2$$

by using Cauchy Schwarz

$$= \sum_i \left(\sum_j a_{ij}^2 \sum_j x_j^2 \right)$$

$$= \left(\sum_i \sum_j a_{ij}^2 \right) \sum_j x_j^2$$

$$= \sum_{i,j} a_{ij}^2 \sum_j x_j^2 = \|A\|_F^2 \cdot \|x\|^2$$

10 pts

6. Find all least squares solutions to the system

$$\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases} \equiv \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}; \quad A^T b = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$A^T A x = A^T b$ is $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ so

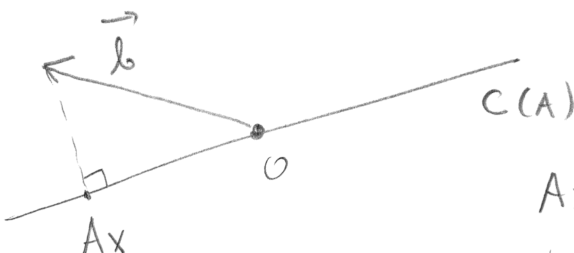
$2x_1 + 2x_2 = 3$ i.e. $x_1 = \frac{3}{2} - x_2$ ← free

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

[OR more pleasantly $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 3/4 \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$]

10 pts

7. The normal equation $A^T Ax = A^T b$ is an expression of perpendicularity of a certain vector to a certain subspace. What vector and subspace are involved? Draw a figure and explain how $A^T Ax = A^T b$ follows.



$Ax - b \perp C(A)$

so $\forall_i$ i^{th} column of A, call it a_i, is $\perp$ to $Ax - b$

i.e. $\forall_i$ $a_i^T(Ax - b) = 0$

so $A^T(Ax - b) = 0$

i.e. $A^T Ax = A^T b$.

6pts

8. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. lengths only!

a) Compute the semi-axis of the ellipse that is the image of the unit circle under the linear transformation T_A of the plane $\mathbf{R}^2$ induced by A.

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$ has char. poly

$$\lambda^2 - \text{trace}\,\lambda + \det = 0$$

$$\lambda^2 - 3\cdot\lambda + 1 = 0$$ with roots $$\lambda_{1,2} = \frac{3 \pm \sqrt{9-4\cdot 1}}{2}$$

Singular values: $$\sigma_1 = \sqrt{\lambda_1} = \sqrt{\frac{3+\sqrt{5}}{2}}$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{\frac{3-\sqrt{5}}{2}}$$ are the semi-axis lengths

3pts

b) What is the value of the operator norm $||A|| := \max_{||x||=1} ||Ax||$? (Here $||x||$ is the ordinary Euclidean norm.)

$$||A|| = \sigma_1 = \sqrt{\frac{3+\sqrt{5}}{2}}.$$

1pt

c)* Sketch the image of the square $\{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$ under T_A and guess-sketch the ellipse (from a)). mark the semi-axes