Diophantine and Minimal but Not Uniquely Ergodic (Almost)

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Abstract

We demonstrate that minimal non-uniquely ergodic behavior can be generated by slowing down a simple harmonic oscillator with diophantine frequency, in contrast with the known examples where the frequency is well approximable by the rationals. The slowing is effected by a singular time change that brings one phase point to rest. The time one-map of the flow has uncountably many invariant measures yet every orbit is dense, with the minor exception of the rest point.

1 Introduction

A discrete dynamical system is given by a homeomorphism $f : X \to X$ of a compact metric space. f is *minimal* iff there is only one non-empty closed invariant subset, the X itself; equivalently, the orbit $\{f^n(x) : n \in \mathbb{Z}\}$ is dense in X for any $x \in X$. ($\mathbb{Z}$ are the integers.) f is *uniquely ergodic* iff there is only one invariant (Borel) probability measure; equivalently, given any $x \in X$, the normalized Dirac measures on the orbit pieces, $\frac{1}{n}\left(\delta_x + \ldots + \delta_{f^{n-1}(x)}\right)$, weak*-converge to a measure μ that does not depend on x. (See [29, 14].)

Since spaces X of interest are infinite, even uncountable, the concepts of minimality and unique ergodicity retain their appeal with an allowance for a finite *exceptional set*. We call f ***essentially minimal*** iff there is a finite set $E \subset X$ such that X is the only non-empty closed invariant subset that is not contained in E.

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Likewise, f is *essentially uniquely ergodic* iff there is a finite set $E \subset X$ such that there is only one invariant probability measure whose support is not contained in E.

A good example that is both minimal and uniquely ergodic is given by the rotation $f_\alpha : x \mapsto x + \alpha$ on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ where $\mathbb{R}$ are the reals, $\alpha \in \mathbb{R}$ is irrational, and the addition is modulo 1.

Viewed through the prism of Boltzmann's Ergodic Hypothesis (postulating equidistribution of orbits [27]), it is an important early lesson of ergodic theory that unique ergodicity implies minimality on the support of the measure but not vice versa: it is possible that every orbit fills X densely but there are orbits with different limiting probability measures.

A textbook example (see II.7 in [18] or 12.6 in [14]) is a skew product on the two-torus $\mathbb{T}^2 := \mathbb{T} \times \mathbb{T}$ over an irrational circle rotation due to Furstenberg [9]. Related smooth examples (with some extra properties) can be found in [8, 30]. After the initial impetus of [28], plenty of non-uniquely ergodic yet minimal behavior has been uncovered in billiard flows (see e.g. [20, 5]). In symbolic dynamics, unique ergodicity is rare without postulating some extra structure and, by now, we understand that the properties of the set of all invariant measures are quite independent of the topological hypothesis of minimality [25, 6, 17].

On the other hand, minimality and unique ergodicity go hand in hand in many systems of geometric or algebraic origin (e.g. group translations and horocycle flows, see §6.6 in [29] and [10, 19, 21]) conspiring with the relative complexity of the non uniquely ergodic and minimal examples to create a perception that it takes a certain pathology to break that relationship. For instance, all smooth examples contain an element of *fast approximation* based on *Liouville* irrational numbers, which are unusually well approximated by rationals and constitute a measure zero subset of $\mathbb{R}$. The irrationals in the complement comprise *Diophantine* numbers, which resist fast rational approximation and are associated with unique ergodicity, stability, and other good dynamical properties. (See [11] as well as [7, 13] for an introduction and more perspective.)

Our modest goal is to produce a rather simple smooth example of a different kind, one using irrational α of *constant type*, i.e., such that there is $c > 0$ for which $|\alpha - \frac{p}{q}| \geq \frac{c}{q^2}$ for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. (Equivalently, the coefficients a_k in the continued fraction expansion (13) are bounded [24].) Although the α of *constant type* form a set of measure zero in $\mathbb{R}$, by virtue of being the worst approximable Diophantine numbers and including all quadratic irrationals they provide a testing ground for theoretical and numerical exploration.

To introduce a "pathology", we rely on the technique of *singular time change* (going back at least to [22]) whereby a linear flow in an irrational direction on $\mathbb{T}^2$ is slowed down so that one point is brought to rest. This rest point yields a fixed point of the time-one-map f but otherwise the dynamics of f proceed with some non-zero average speed densely filling $\mathbb{T}^2$; f is essentially minimal. We show how to arrange that f is not essentially uniquely ergodic and, measure theoretically, looks much like Furstenberg's example. Unlike Furstenberg's and many other examples that are real-analytic, we only manage C^∞ smoothness on the complement of the rest point.

Theorem 1. *Let $\mathbb{T}^2 := \mathbb{T} \times \mathbb{T}$ be the two dimensional torus and α be an irrational of constant type. There is a continuous function $\Phi : \mathbb{T}^2 \to [0, \infty)$, which has its only zero at $(0,0)$ and is C^∞-smooth on the complement of $(0,0)$, such that the time-one-map $f : \mathbb{T}^2 \to \mathbb{T}^2$ of the flow associated to the system*

$$\begin{cases} \dot{x_1} = \Phi(x_1, x_2)\alpha \\ \dot{x_2} = \Phi(x_1, x_2) \end{cases} \tag{1}$$

is essentially minimal but is not essentially uniquely ergodic. Precisely, the orbit $\{f^n(x) : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}^2$ for $x \neq (0,0)$ yet the natural f-invariant measure (that is absolutely continuous with respect to the area) decomposes into uncountably many ergodic measures, each measurably equivalent to the ordinary length measure on the circle acted upon by the irrational rotation f_α. Additionally, f is topologically mixing.

While toral flows with a rest point at which smoothness is lost arise naturally in generic Hamiltonian systems [2, 26, 16], the specific form of our singularity at $(0,0)$ (see (11) ahead) does not appear in any applicable models that we know of. It remains to be seen in what setting and for what class of α and Φ the phenomenon illustrated by our example occurs in a meaningful way.

Let us outline the construction. We think of the torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$ as $\mathbb{R}^2/\mathbb{Z}^2$. All the flow orbits on $\mathbb{T}^2$ apart from the ones contained in the line $(0,0) + \mathbb{R}(\alpha, 1)$ (with addition modulo one) return repeatedly to the circle $\Sigma := \mathbb{T} \times \{1/2\} \subset \mathbb{T}^2$. Taking Σ as a Poincaré cross-section, they correspond to the orbits of the *special flow* over the rotation f_α and under some *return time function* $r : \mathbb{T} \setminus \{0\} \to (0, \infty)$; see [3]. The plan is to first construct a suitable special flow and then realize it by a vector field.

To be specific, upon setting $K = \{n\alpha : n \in \mathbb{Z}\} \subset \mathbb{T}$, let us realize the special flow as the factor of the flow F^t on $(\mathbb{T} \setminus K) \times \mathbb{R}$ given by $F^t(x, y) \mapsto (x, y + t)$ where we quotient by the $\mathbb{Z}$-action generated by $D : (x, y) \mapsto (x + \alpha, y - r(x))$. (The region above the x-axis and under the graph of $x \mapsto r(x)$ is a fundamental domain, see Figure 1; and the point (x, y) in that domain will correspond to the point on $\mathbb{T}^2$ obtained by flowing $(x - \alpha/2, 1/2) \in \Sigma \subset \mathbb{T}^2$ for time y.)

The crux of our construction is a suitable choice of the function r. (Finding the speed function Φ, realizing that r, is then not hard.) To this end, we fix $1/2 < p < 1$, $b > 2$, $a > 0$, and consider the following function on $\{z \in \mathbb{C} : |z| \leq 1\} \setminus \{1\}$,

$$R(z) := a \left(\log \frac{b}{1 - z} \right)^{1-p}. \tag{2}$$

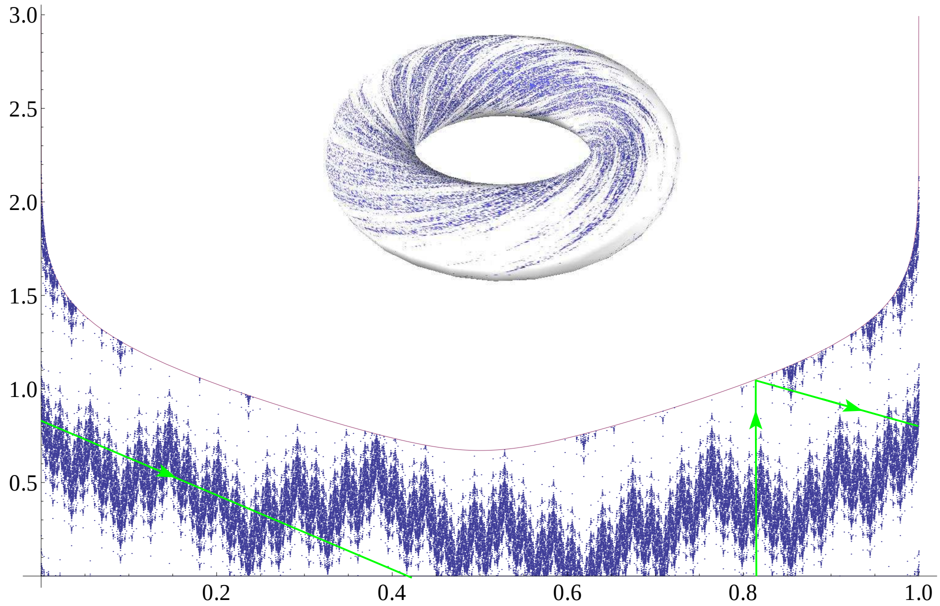

Figure 1: At the bottom, the special flow moves vertically with unit speed until it reaches the graph $y = r(x)$, which is identified with the base $[0, 1]$ after the horizontal translation by $\alpha = (\sqrt{5} - 1)/2 \approx 0.618 \pmod 1$. The jagged pattern is formed by 10^5 points of a dense orbit of the time-one-map and approximates the ergodic measure carried on the graph of the L^2 function $y = h(x)$ (wrapping through the top and bottom). At the top, the 10^5 iterates rendered on the torus by first sending (x, y) to $(x+\alpha y/r(x), y/r(x)) \pmod 1$ and following with the standard embedding of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ into $\mathbb{R}^3$. (Mathematica 8.0 computation with the initial point $(0.4, 0.4)$, $a \approx 0.956$, $b = 3$, $p = 0.52$, and internal precision to 12 decimal places.)

Here we use the cut along the negative real axis in the complex plane $\mathbb{C}$ and the standard branches of the logarithm and the power function, $\log(re^{i\theta}) = \log r + i\theta$ and $(re^{i\theta})^{1-p} = r^{1-p}e^{i(1-p)\theta}$ with $r > 0$ and $\theta \in (-\pi, \pi)$. This makes R analytic on an open set containing $\{|z| \leq 1\} \setminus \{1\}$. In particular, we have a well defined 1-periodic real-analytic function on $\mathbb{R} \setminus \mathbb{Z}$ given by

$$r(x) := \operatorname{Re}\left(R(e^{i2\pi x})\right) \quad (x \in \mathbb{R} \setminus \mathbb{Z}). \tag{3}$$

One can further check that this r is positive, even, U-shaped on its period interval $(0, 1)$, and has the asymptotics as x converges to 0 given by

$$r(x) \sim a \operatorname{Re}\left(\left(\log \frac{b}{2\pi i x}\right)^{1-p}\right) \sim (-\log |x|)^{1-p}. \tag{4}$$

(This should be compared with the purely logarithmic asymptotics $r(x) \sim -\log |x|$ in [2, 26, 16].) Let us adjust the parameter a so that

$$\int_0^1 r(x)\, dx = 1.$$

The special flow under r, of which we think now as a function on $\mathbb{T}\setminus K \subset \mathbb{T}\setminus\{0\}$, has an obvious invariant probability measure μ_{flow} equal to the area $|dx|\otimes ds$ when lifted to $\mathbb{T}\setminus K\times\mathbb{R}$. We want the measure to be non-ergodic and decompose into uncountably many measures carried on (vertically) translated copies of an invariant graph of some function over $\mathbb{T}\setminus K$. To this end, we consider a measurable function $h:\mathbb{T}\setminus K\to\mathbb{R}$ and the family of graphs

$$\Gamma_\sigma := \{(x,y)\in\mathbb{T}\setminus K:\ y = h(x)+\sigma\},\quad \sigma\in[0,1).$$

One way to guarantee invariance under the time-one-map is to request that

$$D(\Gamma_\sigma+(0,1)) = \Gamma_\sigma, \tag{5}$$

which amounts to asking that the following cohomological equation is satisfied

$$h(x+\alpha)-h(x) = 1-r(x)\quad (x\in\mathbb{T}\setminus K). \tag{6}$$

We seek a solution $h\in L^2(|dx|)$, as then $h(x)$ is defined for $|dx|$-almost all $x\in\mathbb{T}$ and each Γ_σ can be equipped with the probability measure μ_σ obtained by lifting $|dx|$ via the projection $(x,y)\mapsto x$. In view of (5), this projection conjugates the time-one-map restricted to the invariant set given by Γ_σ with the irrational rotation f_α. In particular, the μ_σ are ergodic and thus constitute the ergodic decomposition of μ_{flow},

$$\mu_{\text{flow}} = \int_0^1 \mu_\sigma d\sigma.$$

To solve (6), we seek h in the form $h(x)=\operatorname{Re}(H(e^{i2\pi x}))$ where

$$H(e^{i2\pi\alpha}z)-H(z) = 1-R(z). \tag{7}$$

Upon representing $R(z)$ and $H(z)$ by their Taylor series

$$R(z)=\sum_{n=0}^{\infty} R_n z^n \quad\text{and}\quad H(z)=\sum_{n=1}^{\infty} H_n z^n,$$

(7) yields the well known explicit formula

$$H_n = \frac{-R_n}{e^{i2\pi n\alpha}-1}\quad (n=1,2,\ldots). \tag{8}$$

The presence of the "small denominators" leaves it to be seen that these H_n are indeed Taylor coefficients of an analytic function H on the unit disk and that H has a.e. defined and square integrable boundary values on the unit circle, thus yielding an L^2-solution h of (6). The following lemma resolves this difficulty.

Lemma 1. $\sum_n |H_n|^2<\infty$.

The proof of the lemma, in Section 2, is the heart of this note. It uses the continued fraction expansion of α together with the following asymptotics from Theorem 2.31 in [31],

$$\lim_{n\to\infty} \frac{R_n}{\left(\frac{1-p}{n(\log n)^p}\right)} = a. \tag{9}$$

Once we construct the special flow model of our example, we have to present it as a flow of the system (1) by making a function Φ that realizes the function r as the return time to Σ. This is rather straightforward and we construct Φ in Section 3. Near its only zero at $(0,0)$, Φ is given by

$$\Phi(x_1, x_2) = s^2 + \psi(x)^2 \tag{10}$$

where (x, s) is a local coordinate in which the leaves of the irrational foliation on $\mathbb{T}^2$ become vertical, $(x_1, x_2) = (x + s\alpha, s)$. The asymptotics near $x = 0$ is

$$\psi(x) \sim \frac{\pi}{r(x)} \sim \frac{\pi}{(-\log|x|)^{1-p}}. \tag{11}$$

We note that this form of Φ readily guarantees existence and uniqueness of the flow of (1) by a variant of the standard criterion [12] because

$$\left|\int_0^{\pm\epsilon} \frac{ds}{\Phi(s\alpha, s)}\right| \sim \int_0^{\pm\epsilon} \frac{ds}{s^2} = +\infty \quad (\epsilon > 0). \tag{12}$$

Finally, in Section 4, we show that the time-one-map f is mixing and essentially minimal. As will be clear from the argument, this is a general property enjoyed by any flow on $\mathbb{T}^2$ that proceeds in a fixed irrational direction and has a single rest point.

2 Number Theory and L^2 estimate

Before proving Lemma 1 recall the continued fraction expansion [15] of an irrational $\alpha \in (0,1)$ and the sequence of partial quotients $p_k/q_k \to \alpha$, $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$,

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}, \qquad \frac{p_k}{q_k} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}. \tag{13}$$

Here, using $\lfloor\cdot\rfloor$ and $\{\cdot\}$ to denote the integer and the fractional part, respectively, we have $a_1 = \lfloor 1/\alpha \rfloor$, $a_2 = \lfloor 1/\alpha_1 \rfloor$ where $\alpha_1 = \{1/\alpha\}$, $a_3 = \lfloor 1/\alpha_2 \rfloor$ where $\alpha_2 = \{1/\alpha_1\}$, etc. To fix attention, assume that $\alpha \in (0.5, 1)$ so that $a_1 = 1$. The p_k and q_k are determined recursively by $p_{k+1} = a_{k+1}p_k + p_{k-1}$ and $q_{k+1} = a_{k+1}q_k + q_{k-1}$ with $p_{-1} = q_0 = 1, p_0 = q_{-1} = 0$. Since $a_k \geq 1$, the q_k grow at least exponentially:

$$q_k \geq 2^{\frac{k-1}{2}} \quad (k \geq 1). \tag{14}$$

We will use *Ostrowski α-numeration* [23, 1] assigning to any $n \in \mathbb{N}$, a sequence $(d_k)_{k=1}^{\infty}$ such that

$$n = \sum_{k=1}^{\infty} d_k q_k, \tag{15}$$

where $d_k \in \{0, 1, \ldots, a_{k+1}\}$ are all eventually zero. The d_k, called *digits*, are most simply determined by the "greedy algorithm" whereas the leading digit, i.e., the $d_k \neq 0$ with the largest k, is the result of dividing into n the largest $q_k \leq n$, $d_k = \lfloor n/q_k \rfloor$. The second leading digit of n is a result of applying the same process to $n - d_k q_k$, etc. (Just like for the decimal expansion.)

For our purposes, it is better to construct (d_k) in terms of the irrational rotation f_α. Cut the circle open to identify it with $[-\alpha, 1-\alpha)$ and consider the two sides of 0, $J_0 = [-\alpha, 0)$ and $J_1 = [0, 1-\alpha)$. Set $x_k := f_\alpha^k(0)$ for $k \in \mathbb{Z}$. Recall that x_{-q_k} get progressively closer to 0 while alternating between J_0 and J_1 as $k = 0, 1, 2, \ldots$. Moreover, the segments J_k with endpoints 0 and x_{-q_k} present the following picture (see e.g. [4]). Applying $f_\alpha^{-q_{k-1}}$ to J_k and $f_\alpha^{-q_k}$ to J_{k-1} interchanges the two segments so that

$$D_k := J_{k-1} \cup J_k = f_\alpha^{-q_{k-1}}(J_k) \cup f_\alpha^{-q_k}(J_{k-1}) \quad (k \geq 1).$$

This *interval exchange map* $T_k : D_k \to D_k$ induces the first return map on $D_{k+1} \subset D_k$ which coincides with $T_{k+1} : D_{k+1} \to D_{k+1}$.

Now, to a point $x \in D_k$, assign the *k-code* $d \in \{0, \ldots, a_{k+1}\}$ such that $T_k^d(x)$ is the first entry into the return domain D_{k+1}. To be precise, for k odd, the complement $D_k \setminus D_{k+1}$ is a train of segments

$$[x_{-q_{k-1}}, x_{-q_{k-1}-q_k}) \cup [x_{-q_{k-1}-q_k}, x_{-q_{k-1}-2q_k}) \cup \ldots \cup [x_{-q_{k-1}-(a_{k+1}-1)q_k}, x_{-q_{k+1}})$$

one mapping to another by $f_\alpha^{-q_k}$, and the last mapping under $f_\alpha^{-q_k}$ into D_{k+1}. Thus $d = 0$ iff $x \in D_{k+1}$ and otherwise d is such that

$$x \in [x_{-q_{k-1}-(a_{k+1}-d)q_k}, x_{-q_{k-1}-(a_{k+1}-d+1)q_k}).$$

For k even, this is still true if we agree that $[a, b)$ stands for the left-closed and right-open segment with the ends a and b, even if $a > b$.

Given $n \in \mathbb{N}$, its digits are obtained by applying the exchange transformations to x_n and recording the k-codes upon the first entry into D_k as follows. Digit d_1 is the 1-code of x_n. Digit d_2 is the 2-code of $x_{n'}$ where $n' := n - d_1 q_1$. Here it is important to note that $n' \geq 0$. Indeed if $n' < 0$, then $x_{n'} \in D_2$ making it closer to 0 than x_{-q_1}; this implies $n' \leq -q_2$, so $n \leq -q_2 + a_2 q_1 < 0$, contrary to $n \in \mathbb{N}$.

Similar arguments can be made in subsequent steps and we can produce d_3 as the 3-code of $x_{n-d_1q_1-d_2q_2}$, etc. The process stops at the step k generating $n'=0$, at which point $n=d_1q_1+d_2q_2+\ldots+d_kq_k$.

Note that if $d_k\neq 0$ then $T_k^{d_k}(x)$ sits in $[x_{-q_{k+1}},x_{-q_{k+1}-q_k})=D_{k+1}\setminus[x_{-q_{k+1}-q_k},x_{-q_k})$ where x is the point with the code d_k considered in the kth step. It follows that the $k+1$-code of $x':=T_k^{d_k}(x)$ cannot be a_{k+2}. That is the digits we constructed satisfy $d_k>0\implies d_{k+1}<a_{k+2}$. This property implies that our expansion coincides with the Ostrowski expansion [1].

We need the following lower bound on the small denominator in terms of the first non-zero digit in the expansion of n.

Fact 1. *If the Ostrowski expansion of $n\in\mathbb{N}$ is $[0\ldots 0d_kd_{k+1}\ldots]$ with $d_k>0$ (for some $k\geq 1$), then*

$$|e^{i2\pi n\alpha}-1|>\frac{1}{q_{k+1}(a_{k+2}+2)}.$$

Proof: By elementary geometry, $|e^{i2\pi n\alpha}-1|\geq|x_n|$. The construction of the kth digit d_k ensures $x_n\in D_k\setminus D_{k+1}$, so $|x_n|\geq|x_{q_{k+1}}|$ and it suffices to show that

$$|x_{q_{k+1}}|>\frac{1}{q_{k+1}(a_{k+2}+2)}.$$

This is well known and best seen from the fact that

$$J_k,f_\alpha(J_k),\ldots,f_\alpha^{q_{k+1}-1}(J_k),\ \ J_{k+1},f_\alpha(J_{k+1}),\ldots,f_\alpha^{q_k-1}(J_{k+1})$$

cover the circle [4]. Indeed, $|x_{q_{k+1}}|$ is the length of J_{k+1} and, J_k being a union of J_{k+2} together with a_{k+2} translated copies of J_{k+1}, we have

$$|J_k|\leq a_{k+2}|J_{k+1}|+|J_{k+2}|\leq(a_{k+2}+1)|J_{k+1}|.$$

From the covering,

$$q_{k+1}(a_{k+2}+1)|J_{k+1}|+q_k|J_{k+1}|>1,$$

which readily gives the desired inequality $|J_{k+1}|>\frac{1}{q_{k+1}(a_{k+2}+2)}$. $\square$

We are ready to prove Lemma 1. We assume that $\alpha\in(1/2,1)$ is an irrational of constant type, i.e.,

$$C_2:=\frac{1}{\max_k a_k+2}$$

is a positive constant.

Denote by N_k all natural n with the expansion of the form $[0\ldots 0d_kd_{k+1}\ldots]$ with $d_k>0$. By using the asymptotics (9) for R_n, we find a constant $C_1>0$ and $N>0$ such that

$$R_n<\frac{C_1}{n(\log n)^p}\qquad(n>N).$$

This, followed by Fact 1, allows us to estimate

$$\begin{aligned}
\sum_{n>N} |H_n|^2 &= \sum_{n>N} \left| \frac{R_n}{e^{i2\pi n\beta} - 1} \right|^2 \\
&< \sum_{n>N} \left| \frac{1}{e^{i2\pi n\beta} - 1} \frac{C_1}{n(\log n)^p} \right|^2 \\
&\leq \sum_{k=1}^{\infty} \sum_{m\in N_k} \left| \frac{1}{e^{i2\pi m\beta} - 1} \frac{C_1}{m(\log m)^p} \right|^2 \\
&< \sum_{k=1}^{\infty} \sum_{m\in N_k} \left| \frac{q_{k+1}}{C_2} \frac{C_1}{m(\log m)^p} \right|^2 \\
&\leq \sum_{k=1}^{\infty} \sum_{m\in N_k} \left| \frac{q_k}{C_2^2} \frac{C_1}{m(\log m)^p} \right|^2
\end{aligned}$$

where the last inequality used

$$q_{k+1} = a_{k+1}q_k + q_{k-1} \leq a_{k+1}q_k + q_k < (a_{k+1}+2)q_k \leq \frac{q_k}{C_2}.$$

For each k, we write $N_k = \{m_1, m_2, m_3, \dots\}$ with $m_l < m_{l+1}$. Note that $m_1 = q_k$ and the construction of the expansion gives $m_{l+1} \geq m_l + q_k$ (for all $l \geq 1$). Thus $m_l \geq lq_k$ allowing us to push the estimate a bit further:

$$\sum_{k=1}^{\infty} \sum_{m\in N_k} \left| \frac{C_1 q_k}{C_2^2 m(\log m)^p} \right|^2 \leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left| \frac{C_1 q_k}{C_2^2 l q_k (\log(l q_k))^p} \right|^2.$$

Finally, tacking on (14) brings us to

$$\sum_{n>N} |H_n|^2 \leq \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left| \frac{C_1^2}{C_2^4 l^2 (\log(l 2^{\frac{k-1}{2}}))^{2p}} \right| = \frac{C_1^2}{C_2^4} \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \left| \frac{1}{l^2} \frac{1}{(\log(\frac{l}{\sqrt{2}}) + k\log\sqrt{2})^{2p}} \right|.$$

Here each sum over k is finite by comparison with $\sum_{k=1}^{\infty} \frac{1}{k^{2p}} < \infty$ (due to $p > 1/2$) and so is the whole double sum since

$$\begin{aligned}
\sum_{l\geq 2} \sum_{k=1}^{\infty} \left| \frac{1}{l^2} \frac{1}{(\log(\frac{l}{\sqrt{2}}) + k\log\sqrt{2})^{2p}} \right| &\leq \sum_{l\geq 2} \sum_{k=1}^{\infty} \frac{1}{l^2} \frac{1}{k^{2p}} \frac{1}{(\log\sqrt{2})^{2p}} \\
&= \frac{1}{(\log\sqrt{2})^{2p}} \sum_{k=1}^{\infty} \frac{1}{k^{2p}} \sum_{l\geq 2} \frac{1}{l^2}.
\end{aligned}$$

We have shown then that $\sum_n |H_n|^2 < \infty$, proving Lemma 1.

3 Speed Function

We outline the construction of a speed function Φ that is C^∞-smooth away from $(0,0)$ and realizes the return time function r. We rely on smooth extension techniques and do not know if it is possible to make Φ real analytic.

For ease of notation, consider the $\mathbb{Z}$-covering $\mathbb{T}\times\mathbb{R}\to\mathbb{T}^2$ given by $(x,s)\mapsto(x+s\alpha,s)$, in which the irrational foliation unfurls into vertical lines. The deck transformations are generated by $(x,s)\mapsto(x+\alpha,s-1)$. The lift of the desired $\Phi:\mathbb{T}^2\to[0,\infty)$ can be sought as a $\mathbb{Z}$-equivariant $\tilde\Phi:\mathbb{T}\times\mathbb{R}\to[0,\infty)$ vanishing only over $(0,0)$ and such that

$$\int_{-1/2}^{1/2}\frac{ds}{\tilde\Phi(x,s)}=r(x)\quad(x\in\mathbb{T}\setminus\{0\}).\tag{16}$$

The main issue is realizing the prescribed return time for the points passing near the rest point $(0,0)$ so we will first concentrate attention on the ϵ-square $(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$ centered at $(0,0)$ for some small $\epsilon>0$. In fact, we shall take a bit less than half of the available return time, $r_{\mathrm{loc}}(x):=(r(x)-C)/2$, where $C\in(0,\min_x r(x))$, and construct $\Phi_{\mathrm{loc}}(x,s)$ for $(x,s)\in(-\epsilon,\epsilon)\times(0,\epsilon)$ so that the time of travel through this upper half-square is

$$\int_0^\epsilon\frac{ds}{\Phi_{\mathrm{loc}}(x,s)}=r_{\mathrm{loc}}(x)\quad(0<|x|<\epsilon).\tag{17}$$

One can seek Φ_{loc} in the form

$$\Phi_{\mathrm{loc}}(x,s):=s^2+\psi(x)^2\tag{18}$$

where ψ is a non-negative function with $\psi(0)=0$, yet to be determined.

We want

$$\int_0^\epsilon\frac{ds}{\Phi_{\mathrm{loc}}(x,s)}=\int_0^\epsilon\frac{ds}{s^2+\psi(x)^2}=\arctan\left(\frac{\epsilon}{\psi(x)}\right)\frac{1}{\psi(x)}\sim\frac{\pi}{2\psi(x)},$$

where the asymptotics is for $|x|\to 0$. Because $u\mapsto\arctan(\epsilon u)\,u$ is a homeomorphism of $[0,\infty)$ and C^∞-diffeomorphism of $(0,\infty)$, solving

$$\arctan\left(\frac{\epsilon}{\psi(x)}\right)\frac{1}{\psi(x)}=r_{\mathrm{loc}}(x)\tag{19}$$

for $u=\frac{1}{\psi(x)}$ and then $\psi(x)$ itself represents no problem and yields $\psi(x)$ that is C^∞ at $x\in\mathbb{T}\setminus\{0\}$ and has the asymptotics near $x=0$ given by

$$\psi(x)\sim\frac{\pi}{2r_{\mathrm{loc}}(x)}\sim\frac{\pi}{r(x)}.$$

It remains to extend the Φ_{loc} to $\tilde\Phi$ satisfying (16). First, still just using (18) and (19), Φ_{loc} extends to the square $Q=(-\epsilon,\epsilon)\times(-\epsilon,\epsilon)$. In fact, provided $\epsilon>0$

was chosen small enough, if we take a slightly larger $\epsilon' > \epsilon$, our Φ_{loc} is well defined on the square $Q' := (-\epsilon', \epsilon') \times (-\epsilon', \epsilon')$ and C^∞ away from $(0,0)$.

Now, abusing the notation a bit, we identify Q and Q' with the open subsets in $\mathbb{T}^2$ corresponding to them via the covering map. There are C^∞-smooth $\beta_1, \beta_2 : \mathbb{T}^2 \to [0, \infty)$ such that the support of β_1 is in Q' and $\beta_1|_Q = 1$, the support of β_2 is in $\mathbb{T}^2 \setminus \overline{Q}$, and $\beta_1 + \beta_2 > 0$. The expression (taking $\beta_1 \Phi_{\text{loc}} = 0$ outside Q')

$$\Phi_0 := \beta_1 \Phi_{\text{loc}} + \lambda \beta_2 \quad (\lambda > 0) \tag{20}$$

defines $\Phi_0 : \mathbb{T}^2 \to [0, \infty)$ that coincides with Φ_{loc} on Q and converges uniformly to infinity on $\mathbb{T}^2 \setminus Q'$ as $\lambda \to \infty$. Had we taken $\epsilon' - \epsilon > 0$ small enough, we can then select large $\lambda > 0$ so that $r_0(x) := \int_{-1/2}^{1/2} \frac{ds}{\tilde{\Phi}_0(x,s)}$ (where $\tilde{\Phi}_0$ is the lift of Φ_0) satisfies

$$r_0(x) < r(x) \quad (x \in \mathbb{T} \setminus \{0\}). \tag{21}$$

(Here we used that $2r_{\text{loc}}(x) = r(x) - C$, so $C > 0$ of the desired return time $r(x)$ is unrealized by Φ_{loc}.) Thus the yet unrealized time of travel $r(x) - r_0(x)$ is a positive function and, although a priori C^∞ only at $x \neq 0$, it naturally C^∞-extends to $x = 0$ by virtue of the formula

$$r(x) - r_0(x) = 2r_{\text{loc}}(x) + C - r_0(x) = C - \int_{-1/2}^{-\epsilon} \frac{ds}{\tilde{\Phi}_0(x,s)} - \int_{\epsilon}^{1/2} \frac{ds}{\tilde{\Phi}_0(x,s)} \quad (x \in (-\epsilon, \epsilon)).$$

To finish, pick a C^∞-function $\beta_3 : \mathbb{T}^2 \to [0, \infty)$ that is supported on the annulus A in $\mathbb{T}^2$ given (in the lift) by $\mathbb{T} \times (1/8, 3/8)$ and strictly positive on the circle $\mathbb{T} \times \{2/8\}$. By the implicit function theorem, the equation

$$\int_{1/8}^{3/8} \frac{1}{\tilde{\Phi}_0(x,s) + \eta(x)\tilde{\beta}_3(x,s)} - \frac{1}{\tilde{\Phi}_0(x,s)} \, ds = r(x) - r_0(x) \quad (x \in \mathbb{T}), \tag{22}$$

has a solution $\eta(x)$ that is C^∞ in $x \in \mathbb{T}$. The product $\eta\beta_3$ gives a C^∞-function on $\mathbb{T}^2$ (supported on A) and

$$\Phi := \Phi_0 + \eta\beta_3 \tag{23}$$

is the desired speed function satisfying (16) (as seen by combining (22) and the definition of r_0).

4 Minimality and Mixing

Let (f^t) be a flow whose orbits are solutions to the ODE system (1) in Theorem 1. The goal is to see that its time-one-map $f : \mathbb{T}^2 \to \mathbb{T}^2$, $f := f^1$, is topologically mixing and has only two closed invariant sets, $\mathbb{T}^2$ and $\{(0,0)\}$. We start with the topological mixing, which is a consequence of "snagging" of open sets on the fixed point $(0,0)$ and similar to the "stretching" in [8].

Lemma 2. *f is topologically mixing, i.e., for any non-empty open $U, V \subset \mathbb{T}^2$, there is $n_0 > 0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq n_0$.*

We indicate with tilde objects lifted to the cover $\mathbb{R}^2$ of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. For instance, $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of f. We take it so that $(0,0)$ is fixed, $\tilde{f}(0,0) = (0,0)$.

Proof of Lemma 2. Take $\Delta > 0$ so that V contains some ball of radius Δ. It suffices to find $n_0 > 0$ such that $f^n(U)$ is Δ-dense in $\mathbb{T}^2$ for all $n \geq n_0$.

Define L_- as the set of points that flow in forward time to the stopped point $(0,0)$, and L_+ as the set of points that flow to $(0,0)$ in backward time. $L_\pm$ are half-lines densely immersed in $\mathbb{T}^2$. Take $L_l \subset L_+$ to be the finite sub-arc starting at $(0,0)$ and of length l; and denote by $\tilde{L}_l$ the line segment obtained as the lift of L_l that contains $(0,0) \in \mathbb{R}^2$. Choose l such that L_l is $\Delta/2$-dense in $\mathbb{T}^2$. Also, find an $\varepsilon \in (0, \Delta/2)$ so that the ε-neighborhood of $\tilde{L}_l$ intersects $\mathbb{Z}^2$ only at $(0,0)$.

Since L_- is dense, U contains a point $p_- \in L_-$. We select a large enough m, so that $p_\varepsilon := f^m(p_-) \in f^m(U)$ is in the ε neighborhood of $(0,0)$. Take $\delta_\varepsilon \in (0, \varepsilon)$ small enough so that $U_0 := B_{\delta_\varepsilon}(p_\varepsilon) \subset f^m(U)$. For large n we observe that the open set $U_n := f^n(U_0) \subset f^{m+n}(U)$ is a topological disk shaped like a very elongated letter "U" whose bend winds tightly against the fixed point $(0,0)$ and the *arms* stretch far along L_+. (Figuratively speaking, pushed by the flow for time n, U_0 got "snagged" on $(0,0)$.) The idea is that, there is $n_1 > 0$ so that, if $n > n_1$, the arms are long enough that L_l is contained in their ϵ-neighborhood making the arms (and thus $f^{m+n}(U)$) Δ-dense in $\mathbb{T}^2$ since $\epsilon + \epsilon < \Delta$.

For the reader that is not yet convinced we discuss "snagging" a bit more carefully. We select $p \in U_0 \setminus L_-$ and a continuous path $\gamma \subset U_0$ from p_ε to p. Since, as $n \to \infty$, $f^n(p_\varepsilon)$ converges to $(0,0)$ and f^n flows $p \notin L_-$ by an amount that increases to infinity, we may find an $n_1 > 0$ so that, for the lift $\tilde{p}$ of p near $(0,0)$, $n \geq n_1$ guarantees

$$|(0,0) - \tilde{f}^n(\tilde{p})| > \sqrt{l^2 + \delta_\varepsilon^2}. \tag{24}$$

Now, $f^n(\gamma)$ is a path joining $f^n(p_\varepsilon)$ to $f^n(p)$ that stays within the ε neighborhood of L_+ (because the flow advances in the direction of L_+ and $\gamma \subset U_0$). Inequality (24) and basic geometry imply that the ε-neighborhood of $f^n(\gamma)$ actually contains the arc L_l. By the assumption on L_l and $\epsilon + \epsilon < \Delta$, $f^n(\gamma) \subset f^{m+n}(U)$ is $\Delta = \epsilon + \epsilon$-dense in $\mathbb{T}^2$. That is $f^k(U)$ is Δ-dense in $\mathbb{T}^2$ for $k \geq m + n_1$. $\square$

The almost minimality is established by a general argument used in [8], which we repeat almost verbatim for the reader's convenience. Assume that $X \subset \mathbb{T}^2$ is closed, $f(X) = X$, and $X \neq \{(0,0)\}$. Then, for $t \in [0,1]$, $X_t := \bigcup_{s \in [0,t]}$ is also closed, $f(X_t) = X_t$, and $X_t \neq \{(0,0)\}$. Additionally, X_1 is flow invariant so $X_1 = \mathbb{T}^2$, since $\{(0,0)\}$ is the only other closed flow invariant set. For any $n \in \mathbb{N}$, one can write $\mathbb{T}^2 = X_1 = \bigcup_{k=0}^{n-1} f^{k/n}(X_{1/n})$ to conclude that $X_{1/n}$ has non-empty interior, forcing $X_{1/n} = \mathbb{T}^2$ by the mixing of f. Finally, $X = \bigcap_{n \in \mathbb{N}} X_{1/n} = \mathbb{T}^2$.

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