Masters Comprehensive Exam
Stat 505-506 August 2012

1. Use the linear model $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ with $\boldsymbol{\varepsilon} \sim N(\boldsymbol{0}, \sigma^2\mathbf{V})$.
   For this problem, assume $\boldsymbol{X}$ is of full column rank.

   (a) State the Gauss Markov Theorem.

   (b) What is the (sampling) distribution of $\boldsymbol{X}\widehat{\boldsymbol{\beta}}$? ($\widehat{\boldsymbol{\beta}}$ is the GLS estimate of $\boldsymbol{\beta}$) (10)

2. Suppose we are fitting a four-level treatment and a linear continuous covariate, x, allowing slopes to change with treatment. Assume errors are iid $N(0, \sigma^2)$ and that we have five observations per treatment group with a reasonable range of x values within each of the groups.

   (a) Write out the "treatment effects" model using Greek letters for all parameters. Use β_0 for overall intercept, τ_i's for treatment effects, β_1 for overall slope, and γ_i's for the interaction terms.

   (b) Give an example of a non-estimable function involving a γ_i. Explain why it is not estimable.

   (c) How do we work around the estimability problem to answer questions of interest about the slopes?

3. Suppose that we have a multilevel model which we represent as:

$$\boldsymbol{y}_i = \boldsymbol{X}_i\boldsymbol{\beta} + \boldsymbol{Z}_i b_i + \boldsymbol{\varepsilon}_i, i = 1, 2, ..., 12$$

   where $\boldsymbol{\varepsilon}_i \sim iid\ N(0, \sigma^2\boldsymbol{I})$ independent of $b_i \sim iid\ N(0, \sigma_b^2)$ and each response vector $\boldsymbol{y}_i$ contains 4 values, $\boldsymbol{X}_i$'s are 4 by 2, and each $\boldsymbol{Z}_i$ is a vector of 4 ones.

   (a) What is the correlation between two responses in the same $\boldsymbol{y}_i$ vector?

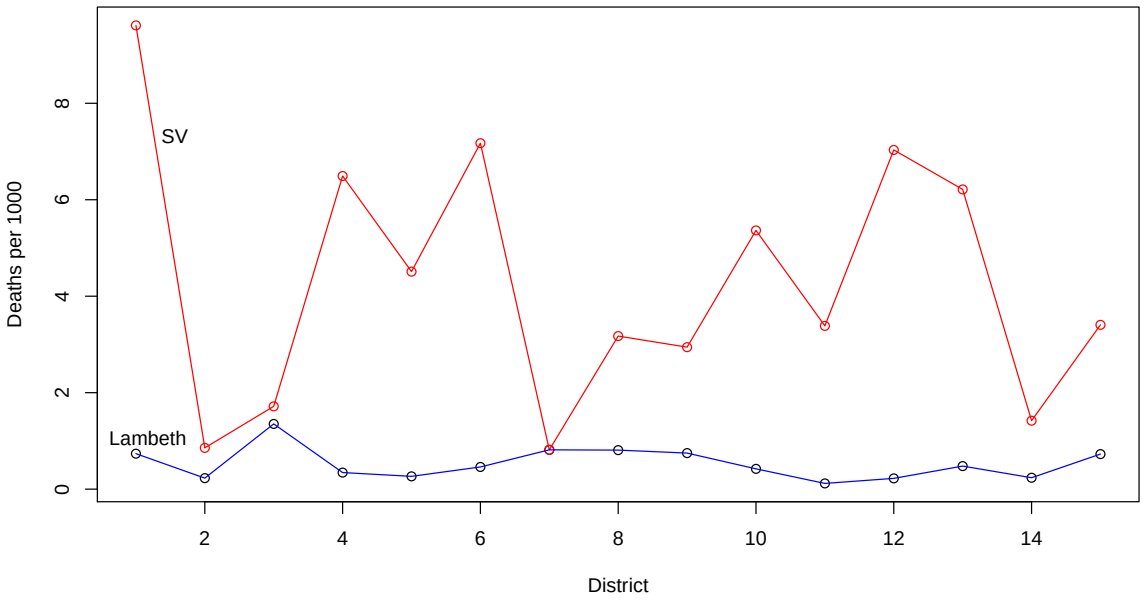
   (b) What is the distribution of $\boldsymbol{y}_i$?

   (c) What is the BLUE of $\boldsymbol{\beta}$?

4. John Snow collected data on cholera deaths in London in 1849 and traced the source of water for houses in which a death occurred to get the following table where "population" is an estimate of the number of people in the district who are supplied water by South & Vauxhall or by Lambeth water companies.

	Water Source			
	South & Vauxhall		Lambeth	
District	population	deaths	population	deaths
Christchurch	9613	11	6409	13
Kent-road	10876	52	7250	5
Borough-road	9517	61	6345	7
London-road	10702	21	7134	8
Trinity	12553	52	8369	6
St. Peter Walworth	17917	84	11944	4
St. Mary	8420	19	5613	1
Waterloo-road 1st	8453	8	5635	2
Waterloo-road 2nd	11009	25	7339	8
Lambeth Church 1st	11045	6	7364	9
Lambeth Church 2nd	16070	34	10714	13
Kennington 1st	14557	63	9704	5
Kennington 2nd	11309	34	7539	3
Brixton	8766	5	5844	2
St. George	9509	34	6340	4

The Lambeth company drew water from the Thames above the points where raw London sewage was discharged into the river, but S&V company's intake was lower. The companies both served each of these districts. Death rates seem quite different:



The data were stacked into a new data frame containing district, water source, population and deaths (4 columns) and the following model was fit:

```
grandDF$surv <- grandDF$pop - grandDF$deaths
grand.fit2 <- lmer(cbind(deaths, surv) ~ water + (1|dist),
                         data = grandDF, family=binomial)
```

and these output were obtained:

```
 Generalized linear mixed model fit by the Laplace approximation
Formula: cbind(deaths, surv ) ~ water + (1 | dist)
   Data: grandDF
   AIC   BIC logLik deviance
 125.4 129.7 -59.72    119.4
Random effects:
 Groups Name        Variance Std.Dev.
 dist   (Intercept) 0.26261  0.51245
Number of obs: 30, groups: dist, 15

Fixed effects:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  -7.7001     0.1710  -45.03   <2e-16
waterSV       2.1428     0.1145   18.72   <2e-16
```

(a) What effect does `family = binomial` have? Explain the assumed model.

(b) Interpret the `waterSV` fixed effect. What does this estimate mean about the odds of cholera death?

(c) Interpret the district effects.

(d) Snow was trying to convince the medical establishment (before germ theory was proposed) that cholera was a contagion spread by drinking contaminated water. What assumptions are needed to provide causal inference in this case?