3 STRATIFIED SIMPLE RANDOM SAMPLING

- Suppose the population is partitioned into disjoint sets of sampling units called **strata**. If a sample is selected within each stratum, then this sampling procedure is known as **stratified sampling**.

- If we can assume the strata are sampled independently across strata, then
  
  (i) the estimator of \( t \) or \( \bar{y}_U \) can be found by combining stratum sample sums or means using appropriate weights
  
  (ii) the variances of estimators associated with the individual strata can be summed to obtain the variance an estimator associated with the whole population. (Given independence, the variance of a sum equals the sum of the individual variances.)

- (ii) implies that only within-stratum variances contribute to the variance of an estimator. Thus, the basic motivating principle behind using stratification to produce an estimator with small variance is to partition the population so that units within each stratum are as similar as possible. This is known as the **stratification principle**.

- In ecological studies, it is common to stratify a geographical region into subregions that are similar with respect to a known variable such as elevation, animal habitat type, vegetation types, etc. because it is suspected that the \( y \)-values may vary greatly across strata while they will tend to be similar within each stratum. Analogously, when sampling people, it is common to stratify on variables such as gender, age groups, income levels, education levels, marital status, etc.

- Sometimes strata are formed based on sampling convenience. For example, suppose a large study region appears to be homogeneous (that is, there are no spatial patterns) and is stratified based on the geographical proximity of sampling units. Taking a stratified sample ensures the sample is spread throughout the study region. It may not, however, lead to any significant reduction in the variance of an estimator.

- But, if the \( y \)-values are spatially correlated (\( y \) values tend to be similar for neighboring units), geographically determined strata can improve estimation of population parameters.

**Notation:**

- \( H \) = the number of strata

- \( N_h \) = number of population units in stratum \( h \) \( h = 1, 2, \ldots, H \)

- \( N = \sum_{h=1}^{H} N_h \) = the number of units in the population

- \( n_h \) = number of sampled units in stratum \( h \) \( h = 1, 2, \ldots, H \)

- \( n = \sum_{h=1}^{H} n_h \) = the total number of units sampled

- \( y_{hj} \) = the \( y \)-value associated with unit \( j \) in stratum \( h \)

- \( \bar{y}_h \) = the sample mean for stratum \( h \)

- \( t_h = \sum_{j=1}^{N_h} y_{hj} \) = stratum \( h \) total

- \( t = \sum_{h=1}^{H} \sum_{j=1}^{N_h} y_{hj} = \sum_{h=1}^{H} t_h \) = the population total

- \( \bar{y}_{hU} = \frac{t_h}{N_h} \) = stratum \( h \) mean

- \( \bar{y}_U = \frac{1}{N} \sum_{h=1}^{H} \sum_{j=1}^{N_h} y_{hj} = \frac{t}{N} \) = the population mean
If a simple random sample (SRS) is taken within each stratum, then the sampling design is called **stratified simple random sampling**.

For stratum \( h \), there are \( \binom{N_h}{n_h} \) possible SRSs of size \( n_h \). Therefore, there are \( \binom{N_1}{n_1} \binom{N_2}{n_2} \cdots \binom{N_H}{n_H} \) possible stratified SRSs for specified stratum sample sizes \( n_1, \cdots, n_H \).

If \( S_{str} \) is a stratified SRS, then the probability of selecting \( S_{str} \) is

\[
P(S_{str}) = \prod_{h=1}^{H} \frac{1}{\binom{N_h}{n_h}} = \frac{1}{\binom{N_1}{n_1} \binom{N_2}{n_2} \cdots \binom{N_H}{n_H}}
\]

Thus, every possible stratified SRS having stratum sample sizes \( n_1, \cdots, n_H \) has the same probability of being selected.

### 3.1 Estimation of \( \bar{y}_U \) and \( t \)

Because a SRS was taken within each stratum, we can apply the estimator formulas for simple random sampling to each stratum. We can estimate each stratum population mean \( \bar{y}_{hU} \) and each stratum population total \( t_h \). The formulas are:

\[
\bar{y}_{hU} = \bar{y}_h = \frac{1}{n_h} \sum_{j=1}^{n_h} y_{hj} \quad \quad \bar{t}_h = N_h \bar{y}_h = \quad \quad \quad \quad (24)
\]

Because each \( \bar{t}_h \) is an unbiased estimator of the stratum total \( t_h \) for \( i = 1, 2, \ldots, k \), their sum will be an unbiased estimator of the population total \( t \). That is,

\[
\bar{t}_{str} = \sum_{h=1}^{H} \bar{t}_h
\]

is an unbiased estimator of \( t \). An unbiased estimator of \( \bar{y}_U \) is a weighted average of the stratum sample means

\[
\bar{y}_{Ustr} = \frac{\bar{t}_{str}}{N} = \frac{1}{N} \sum_{h=1}^{H} N_h \bar{y}_h \quad \text{or, equivalently,} \quad \bar{y}_{Ustr} =
\]

where \( \bar{y}_{Ustr} \) is the weighting factor for stratum \( h \).

Before we can study \( V(\bar{t}_{str}) \) and \( V(\bar{y}_{Ustr}) \), we need to look at the within-stratum variances.

Because a SRS is taken within stratum \( h \), we can apply the results for simple random sampling estimators to each stratum. The variances of the stratified SRS estimators of the mean and total are:

\[
V(\bar{y}_{Uh}) = \quad \quad V(\bar{t}_h) = \quad \quad \quad \quad (25)
\]

where \( S_h^2 = \frac{1}{N_h - 1} \sum_{j=1}^{N_h} (y_{hj} - \bar{y}_{hU})^2 \) is the finite population variance for stratum \( h \).
Because the simple random samples are independent across the strata, the variance of \( \hat{t}_{str} \) is the sum of the individual stratum variances:

\[
V(\hat{t}_{str}) = \sum_{i=1}^{H} V(\hat{t}_h) = \sum_{i=1}^{H}
\]

(26)

- Dividing by \( N^2 \), gives the \( V(\hat{y}_U str) \):

\[
V(\hat{y}_U str) = \left( \frac{1}{N^2} \right) V(\hat{t}_{str}) = \left( \frac{1}{N^2} \right) \sum_{i=1}^{H} N_h(N_h - n_h) \frac{S_h^2}{n_h}
\]

(27)

- Because \( S_h^2 \) is unknown, we use \( s_h^2 \) to get an unbiased estimator of \( V(\hat{t}_h) \):

\[
\hat{V}(\hat{t}_h) = \]

(28)

where \( s_h^2 \) is the sample variance of the \( n_h \) \( y \)-values sampled from stratum \( h \).

- Substitution of (28) into (26) and (27) produce the estimated variances of the stratified SRS estimators:

\[
\hat{V}(\hat{t}_{str}) = \sum_{h=1}^{H} N_h(N_h - n_h) \frac{s_h^2}{n_h} \quad \hat{V}(\hat{y}_U str) = \left( \frac{1}{N^2} \right) \sum_{h=1}^{H} N_h(N_h - n_h) \frac{s_h^2}{n_h}
\]

(29)

- Taking a square root of \( \hat{V}(\hat{t}_{str}) \) or \( \hat{V}(\hat{y}_U str) \) yields the corresponding standard error. This will be used when generating confidence intervals for \( t \) or \( y_U \).

- For the estimated variances of the estimators given in (29), we are assuming that all \( n_h > 1 \) (because \( s_h^2 \) is undefined for \( n_h = 1 \)). Cochran (1977 pages 138-140) discusses two potential methods of dealing with the extreme case where all \( n_h = 1 \).

**Stratification Example with Strong Spatial Correlation**

- Abundance counts for the population in Figures 5a and 5b show a strong diagonal spatial correlation. The region has been gridded into a \( 20 \times 20 \) grid of \( 10 \text{ m} \times 10 \text{ m} \) quadrats. The total abundance \( t = 13354 \). This population was stratified in two different ways:

  (i) Into the four \( 10 \times 10 \) strata shown in Figure 5a.

  Stratum sizes are \( N_h = 100 \) and stratum sample sizes are \( n_h = 5 \) for \( h = 1, 2, 3, 4 \).

  Stratum sample totals \( \sum_{j=1}^{n_h} y_{hj} \) are 124, 158, 172, and 223 for \( h = 1, 2, 3, 4 \).

  Stratum sample means \( \bar{y}_h \) are 24.8, 31.6, 34.4, and 44.6 for \( h = 1, 2, 3, 4 \).

  Stratum sample variances are \( s_1^2 = 21.7, s_2^2 = 13.3, s_3^2 = 45.3, \) and \( s_4^2 = 41.3 \).
(ii) Into seven unequal size diagonally-oriented strata shown in Figure 5b.

Stratum sizes are \( N_1 = N_7 = 45 \), \( N_2 = N_6 = 60 \), \( N_3 = N_5 = 66 \), and \( N_4 = 58 \).

Stratum sample sizes are \( n_1 = n_7 = 3 \), \( n_2 = n_5 = n_6 = 5 \), and \( n_4 = 4 \).

Stratum sample totals \( \sum_{j=1}^{nh} y_{hj} \) are 65, 122, 153, 178, 203, and 143 for \( h = 1, 2, 3, 4, 5, 6, 7 \), respectively.

Stratum sample means \( \bar{y}_h \) are 21.6, 24.4, 30.6, 35.75, 35.6, 40.6, and 47.6 for \( h = 1, 2, 3, 4, 5, 6, 7 \), respectively.

Stratum sample variances are \( s^2_1 = 10.3 \), \( s^2_2 = 14.8 \), \( s^2_3 = 19.3 \), \( s^2_4 = 4.25 \), \( s^2_5 = 8.3 \), \( s^2_6 = 10.8 \), and \( s^2_7 = 26.3 \), respectively.

- For the stratified SRSs in Figure 5a and Figure 5b:
  - Calculate \( \hat{t}_{str} \), \( \hat{\bar{y}}_{U,str} \), and their standard errors.
  - Calculate 95% confidence intervals for \( t \) and \( \bar{y}_U \).

### 3.1.1 Confidence Intervals for \( \bar{y}_U \) and \( t \)

- If all of the stratum sample sizes \( n_h \) are sufficiently large (Thompson suggests \( n_h \geq 30 \)), approximate 100(1 – \( \alpha \))% confidence intervals for \( \bar{y}_U \) and \( t \) are

\[
\hat{\bar{y}}_{U,str} \pm z^* \sqrt{\hat{V}(\hat{\bar{y}}_{U,str})} \quad \quad \hat{t}_{str} \pm z^* \sqrt{\hat{V}(\hat{t}_{str})} \tag{30}
\]

where \( z^* \) is the upper \( \alpha/2 \) critical value from the standard normal distribution.

- For smaller sample sizes, the following confidence intervals have been recommended:

\[
\hat{\bar{y}}_{U,str} \pm t^* \sqrt{\hat{V}(\hat{\bar{y}}_{U,str})} \quad \quad \hat{t}_{str} \pm t^* \sqrt{\hat{V}(\hat{t}_{str})} \tag{31}
\]

where \( t^* \) is the upper \( \alpha/2 \) critical value from the \( t(d) \) distribution. In this case, \( d \) is Satterthwaite’s (1946) approximate degrees of freedom \( d \) where

\[
d = \frac{\left( \sum_{h=1}^{H} a_h s^2_h \right)^2}{\sum_{h=1}^{H} (a_h s^2_h)^2/(n_h - 1)} = \frac{(\hat{V}(\hat{t}_{str}))^2}{\sum_{h=1}^{H} (a_h s^2_h)^2/(n_h - 1)} \tag{32}
\]

where \( a_h = N_h(N_h - n_h)/n_h \).

- Lohr (page 79) mentions that some software packages will use \( n - H \) degrees of freedom (instead of the approximate degrees of freedom). Both R and SAS use \( n - H \) as the default degrees of freedom.

- If the stratum sample sizes \( n_h \) are all equal and the stratum sizes \( N_h \) are all equal, then the degrees of freedom reduces to \( d = n - H \) where \( n = \sum n_h \) is the total sample size.

- One-sided confidence intervals can be generated just like those using SRS. Just use \( t^* \) using the upper \( \alpha \) critical value from the \( t(d) \) distribution.
Figure 5a: Equal Size Strata

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Figure 5b: Unequal Size Strata

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<th>4</th>
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### 3.2 Using R and SAS to Analyze a Stratified SRS

Datasets used in the R code

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<th>R dataset from Figure 5a</th>
<th>R dataset from Figure 5b</th>
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<tr>
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<td>60</td>
</tr>
<tr>
<td>42</td>
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</table>

**R code for Stratified SRS (Figure 5a)**

```r
source("c:/courses/st446/r-code/confintt.r")

# t-based confidence intervals for SRS in Figure 5a

library(survey)
strat5adat <- read.table("c:/courses/st446/r-code/fig5a.txt", header=T)
# strat5adat

strat_design <- svydesign(id=~1, fpc=~fpc, strata=~stratum, data=strat5adat)
strat_design

esttotal <- svytotal(~count,strat_design)
print(esttotal,digits=15)
confint.t(esttotal,degf(strat_design),level=.95)
confint.t(esttotal,degf(strat_design),level=.95,tails='lower')
confint.t(esttotal,degf(strat_design),level=.95,tails='upper')

estmean <- svymean(~count,strat_design)
print(estmean,digits=15)
confint.t(estmean,degf(strat_design),level=.95)
confint.t(estmean,degf(strat_design),level=.95,tails='lower')
confint.t(estmean,degf(strat_design),level=.95,tails='upper')
```

50
R output for Stratified SRS (Figure 5a)

(For the population total)

---
mean( count ) = 13540.00000
SE( count ) = 480.66620
Two-Tailed CI for count where alpha = 0.05 with 16 df
  2.5 %  97.5 %
  12521.03317  14558.96683
---

---
mean( count ) = 13540.00000
SE( count ) = 480.66620
One-Tailed (Lower) CI for count where alpha = 0.05 with 16 df
  5 % upper
  12700.81272 infinity
---

---
mean( count ) = 13540.00000
SE( count ) = 480.66620
One-Tailed (upper) CI for count where alpha = 0.05 with 16 df
  lower 95 %
  -infinity 14379.18728
---

(For the population mean)

---
mean( count ) = 33.85000
SE( count ) = 1.20167
Two-Tailed CI for count where alpha = 0.05 with 16 df
  2.5 %  97.5 %
  31.30258  36.39742
---

---
mean( count ) = 33.85000
SE( count ) = 1.20167
One-Tailed (Lower) CI for count where alpha = 0.05 with 16 df
  5 % upper
  31.75203 infinity
---

---
mean( count ) = 33.85000
SE( count ) = 1.20167
One-Tailed (upper) CI for count where alpha = 0.05 with 16 df
  lower 95 %
  -infinity 35.94797
---
R code for Stratified SRS (Figure 5b)
The R code is exactly the same as the R code for the Figure 5a data analysis except you read in the data file `fig5b.txt`.

R output for Stratified SRS (Figure 5b)

(For the population total)

```
mean( count ) = 13462.70000
SE( count ) = 256.02201
Two-Tailed CI for count where alpha = 0.05 with 23 df
  2.5 %       97.5 %
  12933.07812  13992.32188
```

```
mean( count ) = 13462.70000
SE( count ) = 256.02201
One-Tailed (Lower) CI for count where alpha = 0.05 with 23 df
  5 %  upper
  13023.91117  infinity
```

```
mean( count ) = 13462.70000
SE( count ) = 256.02201
One-Tailed (upper) CI for count where alpha = 0.05 with 23 df
  lower  95 %
  -infinity  13901.48883
```

(For the population mean)

```
mean( count ) = 33.65675
SE( count ) = 0.64006
Two-Tailed CI for count where alpha = 0.05 with 23 df
  2.5 %       97.5 %
  32.33270    34.98080
```

```
mean( count ) = 33.65675
SE( count ) = 0.64006
One-Tailed (Lower) CI for count where alpha = 0.05 with 23 df
  5 %  upper
  32.55978  infinity
```

```
mean( count ) = 33.65675
SE( count ) = 0.64006
One-Tailed (upper) CI for count where alpha = 0.05 with 23 df
  lower  95 %
  -infinity  34.75372
```
Using Proc SurveyMeans in SAS:

- When the stratum unit totals \(N_d\) are known, you must create a variable called \_total\_ that assigns \(N_h\) to each stratum level. It must be called \_total\_. In the following examples, the stratum variable is called Area.

- You also need to create a weight variable which takes on the value \(N_h/n_h\). In the following examples, the weight variable is called W and it appears in the Weight statement.

- Include the option total=(dataname) in the Proc SurveyMeans statement. (dataname) is the name of the data set. In the first example, the dataname is fig\_5a. In the second example, the dataname is fig\_5b.

- Include a Stratum statement that contains the stratum variable.

- In the Var statement, include the response variable \(y\). In these examples, \(y\) is Count.

- If you want one-sided confidence intervals for \(\bar{y}_U\) or \(t\), in the Proc SurveyMeans statement enter lclm or uclm for \(\bar{y}_U\) and lclmsum or uclmsum for \(t\). In the second example, I included all 4 options.

- The list option in the Stratum statement produces a table containing information about each stratum.

Analysis of the Stratified SRS in Figure 5a

data fig5a;
   input Area Count @@;
datalines;
1 18 1 23 1 28 1 25 1 30
2 35 2 30 2 26 2 33 2 34
3 33 3 27 3 30 3 44 3 38
4 47 4 36 4 41 4 53 4 46
;
data fig5a; set fig5a;
   if Area = 1 then _total= 100; *** _total_ = Nh ;
   if Area = 2 then _total= 100;
   if Area = 3 then _total= 100;
   if Area = 4 then _total= 100;
   if Area=1 then W = 100/5; *** W = Nh / nh ;
   if Area=2 then W = 100/5;
   if Area=3 then W = 100/5;
   if Area=4 then W = 100/5;
title1 'Analysis of Stratified SRS in Figure 5a';
proc surveymeans data=fig5a total=fig5a mean clm sum clsum df;
   Stratum Area / list;
   Var Count;
   Weight W;
run;

Analysis of Stratified SRS in Figure 5a

The SURVEYMEANS Procedure

Data Summary

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Stratum Information

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<th>N Obs</th>
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<table>
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<td>12521.0332 14558.9668</td>
</tr>
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Analysis of Stratified SRS in Figure 5b

data fig5b;
  input Area Count @@;
datalines;
  1 18 1 24 1 23
  2 28 2 29 2 21 2 21 2 23
  3 34 3 27 3 35 3 32 3 25
  4 34 4 38 4 37 4 34
  5 32 5 38 5 37 5 38 5 33
  6 37 6 38 6 44 6 44 6 40
  7 42 7 49 7 52
;
data fig5b; set fig5b;
  if Area = 1 then _total_= 45; *** _total_ = Nh ;
  if Area = 2 then _total_= 60;
  if Area = 3 then _total_= 66;
  if Area = 4 then _total_= 58;
  if Area = 5 then _total_= 66;
  if Area = 6 then _total_= 60;
  if Area = 7 then _total_= 45;

  if Area=1 then W = 45/3; *** W = Nh / nh ;
  if Area=2 then W = 60/5;
  if Area=3 then W = 66/5;
  if Area=4 then W = 58/4;
  if Area=5 then W = 66/5;
  if Area=6 then W = 60/5;
  if Area=7 then W = 45/3;

title1 'Analysis of Stratified SRS in Figure 5b';

proc surveymeans data=fig5b total=fig5b mean clm sum clsum df
  lclm uclm lclmsum uclmsum ;
  Stratum Area / list;
  Var Count;
  Weight W;
run;
### Analysis of Stratified SRS in Figure 5b

The SURVEYMEANS Procedure

#### Data Summary

- Number of Strata: 7
- Number of Observations: 30
- Sum of Weights: 400

#### Stratum Information

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<tr>
<th>Stratum Index</th>
<th>Area</th>
<th>Population Total</th>
<th>Sampling Rate</th>
<th>N Obs</th>
<th>Variable</th>
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<td>Count</td>
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<td>8.33%</td>
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<tr>
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<td>45</td>
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<td>3</td>
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</table>

#### Statistics

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<th>Lower 95% One-Sided CL for Mean</th>
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<th>Upper 95% One-Sided CL for Sum</th>
<th>Upper 95% One-Sided CL for Sum</th>
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<tbody>
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#### 3.3 Efficiency of Stratified Simple Random Sampling

- Because the variance formulas for $\hat{\tau}_{str}$ and $\hat{\tau}_{Ustr}$ are determined only from within-stratum variances, the precision of the estimators can be improved by forming strata with small $S_h^2$ values (strata with similar $y$-values within each stratum). We will compare $\hat{V}(\overline{y}_U)$ from a SRS to $\hat{V}(\overline{y}_{str})$ from a stratified SRS.

- The population variance can be rewritten as the weighted sum of within-stratum and between-stratum variabilities:

$$
S^2 = \frac{1}{N-1} \sum_{h=1}^{H} \sum_{j=1}^{N_h} (y_{hj} - \overline{y}_U)^2
= \frac{1}{N-1} \left[ \sum_{h=1}^{H} (N_h - 1)S_h^2 + \sum_{h=1}^{H} N_h(\overline{y}_{hU} - \overline{y}_U)^2 \right]
$$
• By substituting this alternative form of $S^2$ into $V(\overline{y}_U)$ and $V(\overline{y}_{U_{str}})$, it can be shown that:

$$V(\overline{y}_U) - V(\overline{y}_{U_{str}}) = \frac{N - n}{Nn(N - 1)} \left[ \sum_{h=1}^{H} N_h (\overline{y}_{hU} - \overline{y}_U)^2 - \frac{1}{N} \sum_{h=1}^{H} (N - N_h) S_h^2 \right].$$

• If this difference in variances is positive, or, equivalently, if

$$\sum_{h=1}^{H} N_h (\overline{y}_{hU} - \overline{y}_U)^2 > \frac{1}{N} \sum_{h=1}^{H} (N - N_h) S_h^2,$$

then we say that $\overline{y}_{U_{str}}$ is more efficient than $\overline{y}_U$.

• A stratified SRS estimator will be more efficient than the SRS estimator of $\overline{y}_U$ or $t$ if the variability between stratum means is sufficiently large relative to the within-stratum variability. This is what happened with the stratification used in Figures 5a and 5b.

### 3.4 Allocation of Sampling Units

• Given that we have enough resources to allocate $n$ units among the $H$ strata, how do we determine the stratum sample sizes $n_h$?

• **Situation 1**: If all strata are the same size and no prior information is available about the population, a reasonable choice would be to assign equal (or nearly equal) sample sizes to the strata. That is, $n_h \approx \cdots$.

  – Example: Consider the stratified population in Figure 5a. Suppose there are enough resources to take a sample of size $n = 50$. How many samples should be taken for each stratum assuming Situation 1?

• **Situation 2**: If the strata are not all the same size and no prior information is available about the population, a reasonable choice would be to assign sample sizes proportional to the sizes of the strata relative to the population size $N$. That is, $n_h \cdots$. This is known as proportional allocation.

  – Example: Consider the stratified population in Figure 5b. Suppose there are enough resources to take a sample of size $n = 50$. How many samples should be taken for each stratum assuming proportional allocation?

• **Situation 3**: The allocation scheme that minimizes $V(\overline{t}_{str})$ is called optimum allocation and requires

$$n_h = \cdots$$

Because the $S_h^2$ values are unknown, we would need prior estimates (possibly from past data or published studies) to attempt optimum allocation.

  – Example: Consider the stratified population in Figure 5b. Suppose there are enough resources to take a sample of size $n = 50$ and we have prior estimates of $s_1 = 3.2$, $s_2 = 3.8$, $s_3 = 4.4$, $s_4 = 2.1$, $s_5 = 2.9$, $s_6 = 3.3$, and $s_7 = 5.1$. How many samples should be taken for each stratum assuming optimum allocation?
- **Situation 4**: In some cases, if the cost of sampling units varies from stratum to stratum, then the total cost of taking a stratified SRS may determine how to allocate units to strata.

  - Let $c_0$ be the fixed (also called “overhead”) cost of the survey that does not depend on what units are in sample. Let $c_h$ be the cost to sample a unit from stratum $h$.

    The total cost $C$ of the sample will be $C = c_0 + \sum_{h=1}^{H} c_h n_h$.

  - Case I: **For a fixed total cost** $C$, the smallest variance $V(\widehat{y}_{U})$ or $V(\widehat{t})$ is achieved by choosing $n_h$ such that:

    $$n_h = \frac{(C - c_0) N_h S_h / \sqrt{c_h}}{\sum_{h=1}^{H} N_h S_h / \sqrt{c_h}}$$

  - Case II: **For a fixed (specified) variance** $V(\widehat{y}_{U})$, the smallest cost is achieved by first determining the total sample size $n$ such that

    $$n = \frac{\left(\sum_{h=1}^{H} N_h S_h \sqrt{c_h}\right) \left(\sum_{h=1}^{H} N_h S_h / \sqrt{c_h}\right)}{N^2 V + \sum_{h=1}^{H} N_h S_h^2}$$

    where $V$ is the fixed variance specified by the researcher. Then, the stratum sample size $n_h$ for $h = 1, 2, \ldots, H$ is

    $$n_h = \frac{n N_h S_h / \sqrt{c_h}}{\sum_{h=1}^{H} N_h S_h / \sqrt{c_h}}$$

    For a fixed $V(\widehat{t})$, use $V = V(\widehat{t})/N^2$ in the formula.

  - If all of the costs ($c_h$) are the same, then the total sample size formula reduces to

    $$n = \frac{\left(\sum_{h=1}^{H} N_h S_h\right)^2}{N^2 V + \sum_{h=1}^{H} N_h S_h^2}$$

  - Because the $S_h$ values are unknown in either Case I or Case II, we would need prior estimates (possibly from past data or published studies) to attempt optimum allocation.

**Example of Situation 4: Case I**: Suppose there is a fixed total cost $C = $3000 and a fixed overhead cost of $c_0 = $500. Consider the stratification used in Figure 5b. The unit sampling costs are

- $c_1 = $20 per unit from stratum 1
- $c_2 = c_3 = $25 per unit from stratum 2 or 3
- $c_4 = $30 per unit from stratum 4
- $c_5 = c_6 = $35 per unit from stratum 5 or 6
- $c_7 = $40 per unit from stratum 7
Then, using $s_h^2$ as an estimate of $S_h^2$: 

$$n_h = \frac{(C - c_0) N_h s_h / \sqrt{c_h}}{\sum_{h=1}^{H} N_h s_h / \sqrt{c_h}} = \frac{2500 N_h s_h / \sqrt{c_h}}{7657.776}$$

<table>
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<tr>
<th>Stratum</th>
<th>$N_h$</th>
<th>$s_h$</th>
<th>$c_h$</th>
<th>$N_h s_h / \sqrt{c_h}$</th>
<th>$N_h s_h / \sqrt{c_h}$</th>
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</table>

The estimated total cost is $\text{cost} + c_0 = +$500 = requiring sampling units.

**Example of Situation 4: Case II**: Suppose there is a fixed variance of $V = V(\hat{t}) = .35$. Consider the stratification used in Figure 5b. The costs are the same as Case I.

Then, using $s_h$ as an estimate of $S_h$:

$$n = \frac{\left(\sum_{h=1}^{H} N_h s_h / \sqrt{c_h}\right) \left(\sum_{h=1}^{H} N_h S_h / \sqrt{c_h}\right)}{N^2V + \sum_{h=1}^{H} N_h S_h^2} \approx \frac{(7657.776)(260.319)}{(400^2)(.35) + 5254.1} =$$

Then, substitution yields

$$n_h = \frac{n N_h s_h / \sqrt{c_h}}{\sum_{h=1}^{H} N_h s_h / \sqrt{c_h}} = \frac{(37.433) N_h s_h / \sqrt{c_h}}{260.319} \approx N_h s_h / \sqrt{c_h}. $$

<table>
<thead>
<tr>
<th>Stratum</th>
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<th>$s_h$</th>
<th>$c_h$</th>
<th>$N_h s_h / \sqrt{c_h}$</th>
<th>$N_h s_h / \sqrt{c_h}$</th>
<th>$n_h S_h^2 h$</th>
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Thus, the minimum cost to achieve $V$ is $\text{cost} + c_0 = +$500 = requiring a total of sampling units.