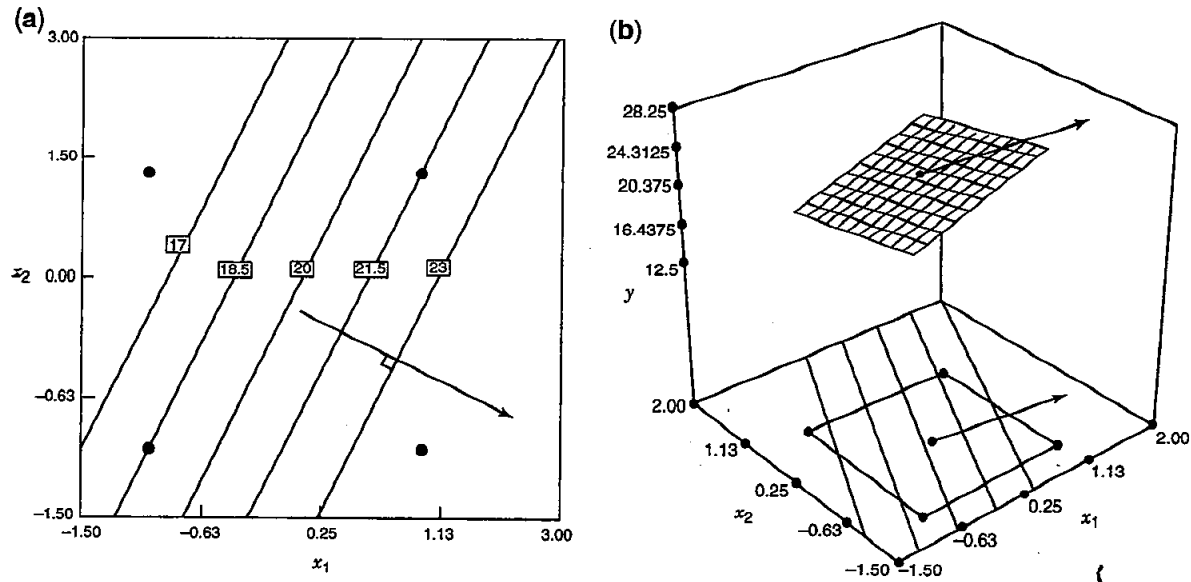


## 6 The Method of Steepest Ascent (Descent)

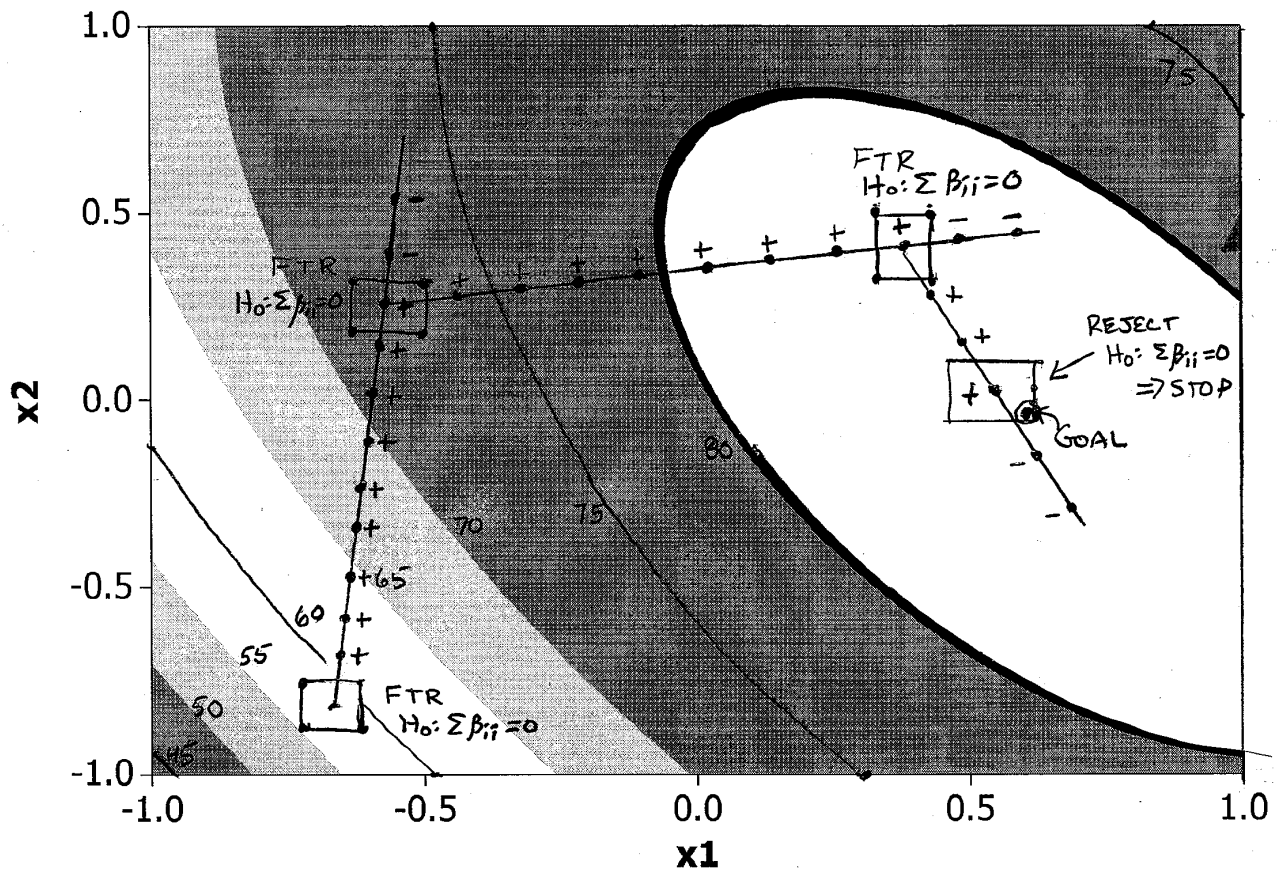
- We assume the researcher is experimenting with a process or production system (perhaps a new process or system).
- Because initial estimates of operating conditions for the system will often be far from the actual optimum, the researcher's objective is to move rapidly to the general vicinity of the optimum via experimentation.
- Therefore, the goal is not to find variable settings which yield an optimum response, but to search for a new operating region in which the process or product is *improved*.
- When the current operating conditions are remote from the optimum, we usually assume a first-order model  $\hat{y} = b_0 + \sum_{i=1}^k b_i x_i$  provides an adequate approximation to the true surface in a small region of the design variables  $x_1, x_2, \dots, x_k$ .
- The **method of steepest ascent** is a method whereby the experimenter proceeds sequentially along the **path of steepest ascent**, that is, along the path of maximum *increase* in the predicted response.
- The **method of steepest descent** is a method whereby the experimenter proceeds sequentially along the **path of steepest descent**, that is, along the path of maximum *decrease* in the predicted response.
- The following steps describe the general procedure:
  1. The experimenter runs an experiment and fits a first-order model  $\hat{y} = b_0 + \sum_{i=1}^k b_i x_i$  in some restricted region of the variables  $x_1, x_2, \dots, x_k$ . The experiment usually contains center-point runs which can be used to perform a lack-of-fit test for curvature.
    - (a) If lack of fit for curvature is not significant, then go to Step 2.
    - (b) If lack of fit for curvature is significant and the experimenter is satisfied that little or no additional information can be obtained using the path of steepest ascent (or path of steepest descent) procedure, then we **stop** applying this method.
  2. Use the fitted first-order model is used to determine a path of steepest ascent (or path of steepest descent).
  3. A series of experimental runs is conducted along the path until no additional increase (or decrease) in response is evident.
  4. Centered near the location along this path which yields a maximum (or minimum) response, a new experiment is designed.
  5. Return to Step 1.
- Once curvature is detected and the method is stopped, a more elaborate experiment to fit a quadratic response surface model should be designed and conducted.
- Note that as the experimental region moves toward the region of optimum operating conditions, it is expected that curvature would be more prevalent (and interactions among the factors would also cease to be negligible). This is why we stop when the lack-of-fit test yields a significant result.
- Because the strategy involves sequential movement from one region to another, several experiments are usually required before stopping. Therefore, design economy is a major concern (that is, you don't want the cumulative number of experimental runs to be too large).

- Hence, only small experimental designs for fitting the first-order model are considered. In particular, two-level factorial and fractional factorial designs (augmented with center points) are typical designs used to fit  $\hat{y} = b_0 + \sum_{i=1}^k b_i x_i$  in a method of steepest ascent or descent problem.



**Figure 5.1** The path of steepest ascent. (a) Contour plot. (b) Three-dimensional response surface view.

### Contour Plot of $y$ vs $x_2, x_1$



## 6.1 Testing for Curvature

- Before discussing the path of steepest ascent or descent, we will review one way to test the adequacy of a first-order model by performing a test for curvature.
- By using two-level factorial and fractional factorial designs (augmented with center points), the experimenter is able to
  1. Obtain an estimate of pure error.
  2. Overall check for interaction effects in the model.
  3. Overall check for quadratic effects (curvature).
- The replicated center points can be used to calculate an estimate of the pure error  $\hat{\sigma}^2 = s^2$  where  $s^2$  is the sample variance of the center point responses.
- The sum of squares for testing for interaction is found by accumulating sum of squares associated with all two-factor interaction terms which can be estimated.
- If there is no curvature present (that is, a plane is an adequate model), then the average response corresponding to the factorial points (say  $\bar{y}_f$ ) should be similar to the average response corresponding to the center points (say  $\bar{y}_0$ ). Thus,  $\bar{y}_f - \bar{y}_0$  is a measure of the overall curvature in the surface over that design space.
- If  $b_{11}, b_{22}, \dots, b_{kk}$  are the coefficients of the  $x_i^2$  terms, then  $\bar{y}_f - \bar{y}_0$  is an estimate of  $b_{11} + b_{22} + \dots + b_{kk}$ .
- Therefore, the sum of squares associated with the contrast  $\bar{y}_f - \bar{y}_0$  is used to test for an overall or 'pure' quadratic effect:

$$SS_{quad} = \quad (7)$$

where  $n_f$  and  $n_0$  are the number of two-level factorial points and  $n_0$  is the number of center points.

- The single degree of freedom sum of squares for testing for curvature is found by comparing  $SS_{quad} = MS_{quad}$  to  $MS_{pure\ error}$ .

Derivation of the  $SS$  for the test for curvature:

- Suppose we have a design with  $n_f$  points from a  $2^k$  (full) or  $2^{k-p}$  (fractional) factorial design plus  $n_0$  center points.
- Note that each of the  $n_f$  factorial points has the form  $(x_1, x_2, \dots, x_k) = (\pm 1, \pm 1, \dots, \pm 1)$  while each of the  $n_0$  center points has the form  $(x_1, x_2, \dots, x_k) = (0, 0, \dots, 0)$ .
- For the second-order model

$$y = \beta_0 + \sum_{i=1}^k \beta_i x_i + \sum_{i < j} \sum_{j=1}^k \beta_{ij} x_i x_j + \sum_{i=1}^k \beta_{ii} x_i^2 + \epsilon,$$

the mean response at a center point is

while the mean response over the set of  $n_f$

factorial points is

because there are an equal number of  $\pm 1$ s for each

factor. Thus, the contrast

is a measure of model curvature.

- To test  $H_0 : \sum_{i=1}^k \beta_{ii} = 0$  against  $H_1 : \sum_{i=1}^k \beta_{ii} \neq 0$ , recall  $SS_{contrast} = \frac{(\text{estimated contrast})^2}{\sum_{i=1}^k c_i^2/n_i}$ , but the estimate of contrast  $\mu_f - \mu_c$  is  $\bar{y}_f - \bar{y}_c$  where

$\bar{y}_f$  is the mean of the responses over the  $n_f$  factorial points, and

$\bar{y}_c$  is the mean of the responses over the  $n_0$  center points. Thus,

$$SS_{quad} = \frac{n_f n_0 (\bar{y}_f - \bar{y}_c)^2}{n_0 + n_f}.$$

## 6.2 Computation of the Path of Steepest Ascent (Descent)

- The movement in  $x_j$  along the path of steepest ascent is proportional to the magnitude of the regression coefficient  $b_j$  with the direction taken being the sign of the coefficient. The path of steepest descent requires the direction to be opposite of the sign of the coefficient.
- Example: If the initial experiment produces  $\hat{y} = 5 - 2x_1 + 3x_2 + 6x_3$ . the path of steepest ascent will result in  $\hat{y}$  moving in a positive direction for increases in  $x_2$  and  $x_3$  and for a decrease in  $x_1$ . Also,  $\hat{y}$  will increase twice as fast for a  $\Delta$  increase in  $x_3$  than for a  $\Delta$  increase in  $x_2$ , and three times as fast as a  $\Delta$  decrease in  $x_1$ .
- Let the hypersphere  $S_r$  be the set of all points of distance  $r$  from the center  $(0, 0, \dots, 0)$  of the initial design region for the coded design variables.
- If  $x_1, x_2, \dots, x_k$  is the set of coded design variables then  $r^2 = \sum_{i=1}^k x_i^2$ .
- The method of steepest ascent assumes that the experimenter wishes to move from the center of the initial design in the direction of that point on  $S_r$  that indicates the **maximum predicted increase in the response**.
- Analogously, method of steepest descent assumes that the experimenter wishes to move from the center of the initial design in the direction of that point on  $S_r$  that indicates the **maximum predicted decrease in the response**.
- Mathematically, the experimenter wishes to find the values of  $x_1, x_2, \dots, x_k$  which maximizes

$$\hat{y} = b_0 + \sum_{i=1}^k b_i x_i \quad \text{subject to the restriction} \quad \sum_{i=1}^k x_i^2 = r^2.$$

- This restricted maximization problem is solved using Lagrange multipliers. That is, we equate to zero the partial derivatives

$$\frac{\partial L(x_1, x_2, \dots, x_k, \lambda)}{\partial x_j} \quad (j = 1, 2, \dots, k) \quad \text{and} \quad \frac{\partial L(x_1, x_2, \dots, x_k, \lambda)}{\partial \lambda}$$

where

$$L(x_1, x_2, \dots, x_k, \lambda) = b_0 + b_1 x_1 + b_2 x_2 + \dots + b_k x_k - \lambda \left( \sum_{i=1}^k x_i^2 - r^2 \right).$$

- Fortunately, Lagrange multiplier problems don't come much easier than this because:

$$\frac{\partial L}{\partial x_j} = \quad \quad \quad (\text{for all } j) \quad \text{and} \quad \frac{\partial L}{\partial \lambda} =$$

- Equating  $\partial L/\partial x_j$  to zero yields

$$x_j = \quad \quad \quad (j = 1, 2, \dots, k). \quad (8)$$

- The procedure is easier to carry out when one selects convenient values of  $\lambda$ , rather than determining the value of  $\lambda$  which corresponds to a particular  $r$ . That is, values are selected which correspond to a particular *increment* in one of the variables.
- Once  $\lambda$  is selected, Equation (8) is used to compute the first point in the path of steepest ascent or path of steepest descent. The experimenter then proceeds to Step 3 of the procedure.
- In practice,  $\lambda$  does not have to be explicitly stated. Usually, the increment is stated for one of the  $x_i$ . Then all other increments are determined.
- For example, note that for  $j = 1$  in (8), we have

$$x_1 = \frac{b_1}{2\lambda} \longrightarrow 2\lambda = \frac{b_1}{x_1}.$$

Substitution into (8) yields

$$x_j = \frac{b_j}{2\lambda} = \quad \quad \quad =$$

Then, if we specify an  $x_1$  value we get the associated  $x_j$  value for  $j = 1, 2, \dots, k$ .

### 6.3 Example Using the Method of Steepest Ascent

From Montgomery (Example 16-1): A chemical engineer is interested in determining the operating conditions that maximize the yield of a process. Two controllable variables influence process yield: reaction time and reaction temperature. She is currently operating the process with a reaction time of 35 minutes and a temperature of 155°F, resulting in yields around 40%. A first-order model will be fit and the method of steepest ascent will be used to find an optimum region.

The engineer determines the initial experimental region should be (30,40) minutes of reaction and (150 – 160)°F. These variables will be coded to  $\pm 1$ . A  $2^2$  design with 5 center points was run yielding:

Natural Variables		Coded Variables		Yield
$\xi_1$	$\xi_2$	$x_1$	$x_2$	$y$
30	150	-1	-1	39.3
30	160	-1	1	40.0
40	150	1	-1	40.9
40	160	1	1	41.5
35	155	0	0	40.3
35	155	0	0	40.5
35	155	0	0	40.7
35	155	0	0	40.2
35	155	0	0	40.6

$$\bar{y}_f = \frac{39.3 + 40.0 + 40.9 + 41.5}{4} =$$

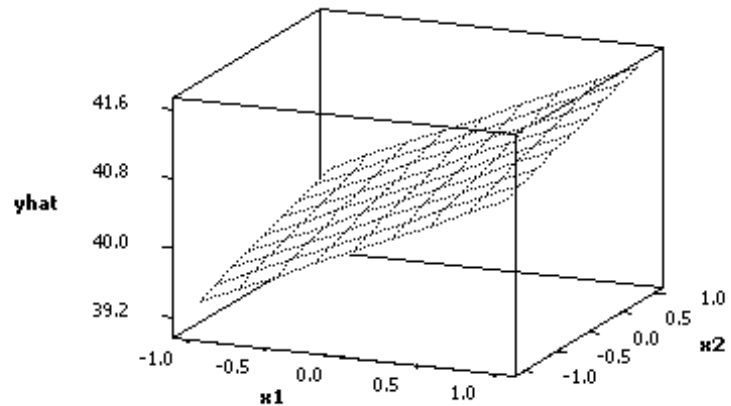
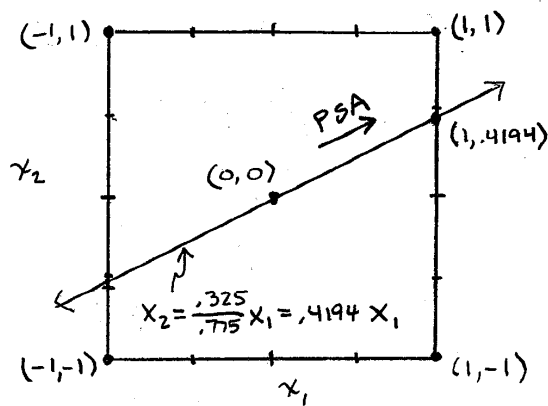
$$\bar{y}_0 = \frac{40.3 + 40.5 + 40.7 + 40.2 + 40.6}{5} =$$

$$SS_{quad} = \frac{n_f n_0 (\bar{y}_f - \bar{y}_0)^2}{n_f + n_0} =$$

$$SS_{pure} = .0172, \quad df_{pure} = 4 \longrightarrow MS_{pure} = .0172/4 = .043$$

$$F_{quad} = \frac{MS_{quad}}{MS_{pure}} = \frac{.00272}{.043} = .0633$$

which has a  $p$ -value of .8137 when compared to an  $F(1,4)$  reference distribution. Note that the curvature test is not significant. Therefore, we continue the next experiment in the path of steepest ascent.



```
DM 'LOG; CLEAR; OUT; CLEAR;';
ODS LISTING;
OPTIONS PS=66 LS=72 NODATE NONUMBER;
```

```
*****;
*** Example 1: PATH OF STEEPEST ASCENT ***;
*****;
```

```
*****;
*** First experiment ***;
*****;
DATA IN; INPUT X1 X2 Y @@;
LINES;
```

```
-1 -1 39.3    -1  1 40.0    1 -1 40.9    1  1 41.5
 0  0 40.3    0  0 40.5    0  0 40.7    0  0 40.2    0  0 40.6
```

```
;
PROC REG DATA=IN;
    MODEL Y = X1 X2 / LACKFIT;
TITLE 'Example 1:  PATH OF STEEPEST ASCENT';
```

```
PROC RSREG DATA=IN;
    MODEL Y = X2 X1 / LACKFIT;
RUN;
```

## Selected Output from PROC REG

Example 1: PATH OF STEEPEST ASCENT

The REG Procedure

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	2.82500	1.41250	47.82	0.0002
Error	6	0.17722	0.02954		
Lack of Fit	2	0.00522	0.00261	0.06	0.9419
Pure Error	4	0.17200	0.04300		
Corrected Total	8	3.00222			

Root MSE	0.17186	R-Square	0.9410
Dependent Mean	40.44444	Adj R-Sq	0.9213
Coeff Var	0.42494		

Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr >  t
Intercept	1	40.44444	0.05729	705.99	<.0001
X1	1	0.77500	0.08593	9.02	0.0001
X2	1	0.32500	0.08593	3.78	0.0092

## Selected Output from PROC RSREG

Example 1: PATH OF STEEPEST ASCENT

The RSREG Procedure

Regression	DF	Type I Sum of Squares	R-Square	F Value	Pr > F
Linear	2	2.825000	0.9410	32.85	0.0033
Quadratic	1	0.002722	0.0009	0.06	0.8137 <--
Crossproduct	1	0.002500	0.0008	0.06	0.8213
Total Model	4	2.830222	0.9427	16.45	0.0095

I used PROC REG to fit the model and also used PROC RSREG to perform the curvature test. Note that the test for curvature is not significant. Therefore, we continue the next experiment in the path of steepest ascent.

## Collecting Data on the Path of Steepest Ascent

- The regression from the first experiment yielded  $\hat{y} = 40.444 + .775x_1 + .325x_2$ .

This path of steepest ascent passes through  $(0, 0)$  and occurs along  $x_2 = .325/.775x_1 \approx .42x_1$ .

- Therefore, the path moves **.325** units in the  $x_2$  direction for every **.775** units in the  $x_1$  direction away from the origin  $(0, 0)$ . Or, rescaled, **.42** units in the  $x_2$  direction for every **1** unit in the  $x_1$  direction away from the origin  $(0, 0)$ .
- The researcher must decide a **basic step size** in one variable. The coding suggests either:  
     5 minutes of reaction time (= 1 unit changed on the coded  $x_1$  scale), or  
     5°F in temperature (= 1 unit changed on the coded  $x_2$  scale).

- Suppose we choose a basic step of 5 minutes of reaction time (denoted  $\Delta x_1 = 1$ ). Therefore, the steps along the path of steepest ascent (PSA) are

$$\Delta x_1 = 1 \quad \Delta x_2 = \frac{b_2}{b_1} \Delta x_1 = \frac{.325}{.775}(1) = \quad \longrightarrow \quad \Delta = (\Delta x_1, \Delta x_2) =$$

- Data will now be collected along the PSA until a decrease in the response is determined (often by observing two consecutive decrease in the response).
- To collect data along the PSA, you will need the transformation back to the original scale:

$$\xi_1 = 35 + 5x_1 \quad \xi_2 = 155 + 5x_2.$$

The following table summarizes the data collection procedure along the PSA:

Point on PSA	$x_1$	$x_2$	Minutes $\xi_1$	°F $\xi_2$	Response $y$	Observed Increase?
$(0, 0)$	0	0	35	155	—	—
$(0, 0) + \Delta$	1	0.42	40	157.1	41.0	—
$(0, 0) + 2\Delta$	2				41.9	
$(0, 0) + 3\Delta$	3				43.1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(0, 0) + 10\Delta$	10					Yes
$(0, 0) + 11\Delta$	11	4.62	90	178.1	79.2	
$(0, 0) + 12\Delta$	12	5.04	95	180.2	78.4	

- Move  $\Delta$  along the PSA from  $(0, 0)$ . This is the point  $(0, 0) + \Delta = (1, .42)$  which, in the original scale is (40 minutes, 157.1°). Collect a response at (40 minutes, 157.1°). The result was  $y = 41.0$ .
- Next, move  $\Delta$  along the PSA from  $(0, 0) + \Delta$  to  $(0, 0) + 2\Delta$ . This new point is  $2 * (1, .42) = (2, .84)$  which, in the original scale is (45 minutes, 159.2°). Collect a response at (45 minutes, 159.2°). The result was  $y = 41.9$  which was larger than  $y = 41.0$  from the previous point along the PSA. Therefore, we continue to the next point on the PSA.
- Now, move  $\Delta$  along the PSA from  $(0, 0) + 2\Delta$  to  $(0, 0) + 3\Delta$ . This new point is  $3 * (1, .42) = (3, 1.26)$  which, in the original scale is (50 minutes, 161.3°). Collect a response at (50 minutes, 161.3°). The result was  $y = 43.1$  which was larger than  $y = 41.9$  from the previous point along the PSA. Therefore, we continue to the next point on the PSA.



- Suppose the response increases along the PSA continues until the 11<sup>th</sup> point. That is, from  $(0,0) + 10\Delta$  to  $(0,0) + 11\Delta$ , the first decrease is observed. This would be from  $(10, 4.20)$  to  $(11, 4.62)$  which, in the original scale is from (85 minutes, 176.0°) to (90 minutes, 178.1°). The response decreased from 80.3 to 79.2.
- One more point is collected along the PSA as further evidence for finding a new path. At  $(0,0) + 12\Delta = (12, 5.04)$  (which, in the original scale is (95 minutes, 180.2°)), a response  $y = 78.4$  was observed. This was smaller than  $y = 79.2$  from the previous point along the PSA. Therefore, we stop collecting data along this path.
- Now, go back to the last point that indicated an increase in  $y$ . This was at  $(x_1, x_2) = (10, 4.20)$ , or,  $(\xi_1, \xi_2) = (85 \text{ minutes}, 176.0^\circ)$ . A new experiment (another  $2^2$  design with center points) is run centered at or near  $(\xi_1, \xi_2) = (85 \text{ minutes}, 176.0^\circ)$ .
- In this case, the experimenter decides the new  $(0,0)$  will be (85 minutes, 175°). The following table contains the results of the second PSA experiment.

Natural Variables		Coded Variables		Yield
$\xi_1$	$\xi_2$	$x_1$	$x_2$	$y$
80	170	-1	-1	76.5
80	180	-1	1	77.0
90	170	1	-1	78.0
90	180	1	1	79.5
85	175	0	0	79.9
85	175	0	0	80.3
85	175	0	0	80.0
85	175	0	0	79.7
85	175	0	0	79.8

- The regression from the second experiment yielded

$$\hat{y} = 79.87 + 1.00x_1 + 0.50x_2.$$

A check for curvature yields a significant  $p$ -value of .0001. Therefore, the first-order model is not adequate, and we believe that we are now in a region containing an optimum response.

- At this point, the experimenters would run a more sophisticated design to isolate the optimum.

```
*****;
*** Second Experiment centered at 85 minutes, 176 degrees ***;
*****;
```

```
DATA INB; INPUT X1 X2 Y @@;
LINES;
-1 -1 76.5    -1  1 77.0    1 -1 78.0    1  1 79.5
 0  0 79.9    0  0 80.3    0  0 80.0    0  0 79.7    0  0 79.8
;
PROC REG DATA=INB;
    MODEL Y = X1 X2 / LACKFIT;
TITLE 'Example 1: Second Experiment centered at 85 minutes, 176 degrees';

PROC RSREG DATA=INB;
    MODEL Y = X2 X1 / LACKFIT;
RUN;
```

## Selected Output from PROC REG

Example 1: Second Experiment centered at 85 minutes, 176 degrees

### The REG Procedure

#### Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	2	5.00000	2.50000	1.35	0.3283
Error	6	11.12000	1.85333		
Lack of Fit	2	10.90800	5.45400	102.91	0.0004
Pure Error	4	0.21200	0.05300		
Corrected Total	8	16.12000			
Root MSE		1.36137	R-Square	0.3102	
Dependent Mean		78.96667	Adj R-Sq	0.0802	
Coeff Var		1.72398			

#### Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr >  t
Intercept	1	78.96667	0.45379	174.02	<.0001
X1	1	1.00000	0.68069	1.47	0.1922
X2	1	0.50000	0.68069	0.73	0.4903

## Selected Output from PROC RSREG

Example 1: Second Experiment centered at 85 minutes, 176 degrees

### The RSREG Procedure

Regression	DF	Type I Sum of Squares	R-Square	F Value	Pr > F
Linear	2	5.000000	0.3102	47.17	0.0017
Quadratic	1	10.658000	0.6612	201.09	0.0001 <--
Crossproduct	1	0.250000	0.0155	4.72	0.0956
Total Model	4	15.908000	0.9868	75.04	0.0005

Note that the quadratic test is significant. Therefore, we stop path of steepest ascent.

## 6.4 Example Using the Method of Steepest Descent

From Montgomery (Problem 16-2): An industrial engineer has developed a computer simulation model of a two-item inventory system. The decision variables are the order quantity and the reorder point for each item. The response to be minimized is total inventory cost. The simulation model is used to produce the data shown in the following table.

Item 1		Item 2						Response
Order Quantity	Reorder Point	Order Quantity	Reorder Point	Coded Levels				Total Cost
$\xi_1$	$\xi_2$	$\xi_3$	$\xi_4$	$x_1$	$x_2$	$x_3$	$x_4$	$y$
100	25	250	40	-1	-1	-1	-1	625
140	45	250	40	1	1	-1	-1	670
140	25	300	40	1	-1	1	-1	663
140	25	250	80	1	-1	-1	1	654
100	45	300	40	-1	1	1	-1	648
100	45	250	80	-1	1	-1	1	634
100	25	300	80	-1	-1	1	1	692
140	45	300	80	1	1	1	1	686
120	35	275	60	0	0	0	0	680
120	35	275	60	0	0	0	0	674
120	35	275	60	0	0	0	0	681

### SAS Code for Example 2:

```
DM 'LOG; CLEAR; OUT; CLEAR;';
ODS LISTING;
OPTIONS PS=66 LS=72 NODATE NONUMBER;

*****;
*** Example 2:  PATH OF STEEPEST DESCENT ***;
*****;

DATA IN; INPUT X1 X2 X3 X4 COST @@;
LINES;
-1 -1 -1 -1 625      1  1 -1 -1 670      1 -1  1 -1 663      1 -1 -1  1 654
-1  1  1 -1 648     -1  1 -1  1 634     -1 -1  1  1 692      1  1  1  1 686
 0  0  0  0 680      0  0  0  0 674      0  0  0  0 681
;
PROC REG DATA=IN;
    MODEL COST = X1 X2 X3 X4 / LACKFIT;
TITLE 'Example 2: PATH OF STEEPEST DESCENT';

PROC RSREG DATA=IN;
    MODEL COST = X1 X2 X3 X4 / LACKFIT;
RUN;
```

## Selected Output from PROC REG

### Example 2: PATH OF STEEPEST DESCENT

#### The REG Procedure

#### Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	4	2541.00000	635.25000	1.74	0.2583
Error	6	2185.18182	364.19697		
Lack of Fit	4	2156.51515	539.12879	37.61	0.0261
Pure Error	2	28.66667	14.33333		
Corrected Total	10	4726.18182			

Root MSE	19.08395	R-Square	0.5376
Dependent Mean	664.27273	Adj R-Sq	0.2294
Coeff Var	2.87291		

#### Parameter Estimates

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr >  t
Intercept	1	664.27273	5.75403	115.44	<.0001
X1	1	9.25000	6.74719	1.37	0.2195
X2	1	0.50000	6.74719	0.07	0.9433
X3	1	13.25000	6.74719	1.96	0.0972
X4	1	7.50000	6.74719	1.11	0.3089

## Selected Output from PROC RSREG

### Example 2: PATH OF STEEPEST DESCENT

#### The RSREG Procedure

Regression	DF	Type I Sum of Squares	R-Square	F Value	Pr > F
Linear	4	2541.000000	0.5376	44.32	0.0222
Quadratic	1	815.515152	0.1726	56.90	0.0171 <--
Crossproduct	3	1341.000000	0.2837	31.19	0.0312
Total Model	8	4697.515152	0.9939	40.97	0.0240

Note that the quadratic test is significant. Therefore, we stop path of steepest descent.

Although we did not need to collect data along a **path of steepest descent (PSD)** for this example, let's still determine what the path would have been.

- Coding:  $x_1 = \frac{\xi_1 - 120}{20}$        $x_2 = \frac{\xi_2 - 35}{10}$        $x_3 = \frac{\xi_3 - 275}{25}$        $x_4 = \frac{\xi_4 - 60}{20}$
- Coding:  $\xi_1 = 120 + 20x_1$        $\xi_2 = 35 + 10x_2$        $\xi_3 = 275 + 25x_3$        $\xi_4 = 60 + 20x_4$
- From the regression, we got

$$\hat{y} = 664.27 + 9.25x_1 + 0.50x_2 + 13.25x_3 + 7.50x_4$$

- Suppose we use  $-25$  units in  $\xi_3$  as the basic step size. Thus,  $\Delta x_3 = -1$ . Then

$$\Delta x_1 = \frac{b_1}{b_3} \Delta x_3 =$$

$$\Delta x_2 = \frac{b_2}{b_3} \Delta x_3 =$$

$$\Delta x_4 = \frac{b_4}{b_3} \Delta x_3 =$$

- The following table summarizes the points on the PSD:

Point	$x_1$	$x_2$	$x_3$	$x_4$	$\xi_1$	$\xi_2$	$\xi_3$	$\xi_4$
$(0, 0)$	0	0	0	0	120	35	275	60
$(0, 0) + \Delta$	-0.698	-.0038	-1	-0.566	106.04	34.62	250	48.68
$(0, 0) + 2\Delta$	-1.396	-.0075	-2	-1.132	92.08	34.25	225	37.36
$(0, 0) + 3\Delta$								
$(0, 0) + 4\Delta$								
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$