Some Basics of Quality Control Charting

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numerical measurement, such as the inside diameter of a piston ring, is to be used to judge the control status of a process, then the variables control chart should be used.

Control charts, whether they be attributes or variables, follow the same general form. Suppose an investigator is concerned with some quality characteristic θ. This quality characteristic may be an attribute, corresponding to a population proportion nonconforming, or the population mean corresponding to a variable. Control charts are graphs which display the realized values of $\hat{\theta}$, an estimator of θ, for each successive sample drawn from the population. The sample numbers are plotted along the horizontal axis and the values of $\hat{\theta}$ are plotted along the vertical axis. Two horizontal lines, called control limits, are drawn on the control chart equidistant from a centerline. Often, two other horizontal lines, called warning limits, will be included on the control chart. These lines are constructed as follows.

Let $\hat{\theta}$ be an estimator of θ based on a random sample of n independent units drawn from an in-control process. Let the mean and standard deviation of the distribution of $\hat{\theta}$ be $\mu_{\hat{\theta}}$ and $\sigma_{\hat{\theta}}$, respectively. Dr. Walter Shewhart proposed a set of formulas to construct control limits for the process characteristic of interest (Montgomery, 1991). The most general form of these control limits is given by:

$$\begin{aligned} &\text{Upper Control Limit(UCL)} = \mu_{\hat{\theta}} + k_1\sigma_{\hat{\theta}} \\ &\text{Centerline} = \mu_{\hat{\theta}} \\ &\text{Lower Control Limit(LCL)} = \mu_{\hat{\theta}} - k_1\sigma_{\hat{\theta}}, \end{aligned} \tag{1}$$

where k_1 is the number of standard deviations a particular value of $\hat{\theta}$ is allowed to vary from $\mu_{\hat{\theta}}$ without signalling an out-of-control process. In essence, control limits provide an upper and lower bound for acceptable values of $\hat{\theta}$.

choose $k_1 = 3$ and $k_2 = 2$. These are suggested values of k_1 and k_2 which have been shown to work well in practice. Certain considerations, such as losses due to producing substandard products, may require the use of smaller values of k_1 and k_2. From here on, $k_1 = 3$ and $k_2 = 2$ will be used.

The time required to perform the sampling and the cost of sampling are two major factors to consider in choosing the sample size. Another consideration is whether or not destructive sampling, sampling which renders the unit unfit for future use, is employed. The sample size must be both small enough to ensure that losses due sampling for the quality control scheme do not exceed the benefits, and large enough to give reasonably accurate results. When the time required to detect a shift in the process needs to be short, sampling more frequently may be more effective in reducing time to detection than increasing the sample size. For example, this may involve taking a sample every half hour rather than every hour.

Control charts are used to test the null hypothesis, H_0, that the process is in control versus the altenative, H_a, that the process is out of control. Various rules have been suggested to decide whether or not to reject H_0. In practice, a subset of these rules can be chosen for implementation. If one or more of the implemented rules is satisfied, reject H_0. A list of seven commonly used rules, along with some of their deviations which make them more sensitive, is given below (Hoyer and Ellis, 1996).

RULE 1: One or more points are plotted outside the control limits. Because the design of the control chart depends on the construction of the control limits, this is the primary rule used by many practitioners, even though it is not the most sensitive in detecting an out-of-control process.

RULE 2: Two out of three consecutive points plot between the warning limits and the control

2.2 Attributes Control Charts

Two types of attributes control charts will be discussed in this paper. The first will be the p-chart for the fraction of units nonconforming. The second will be the c-chart for the number of nonconformities in an inspection unit.

2.2.1 The p-chart

When using the p-chart, each unit of production is classified as either conforming or nonconforming. Therefore, the inspection of one single unit can be viewed as a Bernoulli experiment. Each unit is assumed to have an equal probability of being judged nonconforming. If a sample of n independent units is selected from the same production process and inspected, this then becomes a binomial experiment. Let random variable X be the number of defective units in the sample.

The parameter of interest is p, the fraction nonconforming. The sample fraction nonconforming, $\hat{p}$, should be computed from the data. Let X_i be the number of nonconforming units in the i^{th} sample. By the Central Limit Theorem, for large n, $\hat{p}$ is distributed approximately normal with mean $\mu_{\hat{p}} = p$ and standard deviation $\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$. The control limits are:

$$\begin{aligned} \text{UCL} &= p + 3\sqrt{\tfrac{p(1-p)}{n}} \\ \text{Centerline} &= p \\ \text{LCL} &= p - 3\sqrt{\tfrac{p(1-p)}{n}}. \end{aligned} \tag{3}$$

The construction of the warning limits is similar.

These formulas are based on a known value of p. This value may be specified by management or may be known from extensive research. Quite often, the value of p will not be points can be plotted to see if they plot in control. If so, accept the limits as valid.

Once valid control limits have been computed, process control testing can proceed. Samples should be collected from the same process. Compute the value of $\hat{p}$ for each sample as the data becomes available. Plot the most current value of $\hat{p}$ on the control chart and use a subset of the rules discussed earlier in this paper to determine if the process is running in control.

2.2.2 The c-chart

The c-chart is based on the total number of nonconformities in an inspection unit, where an inspection unit, in general, refers to the sample drawn from a process. A nonconformity, in this case, is defined to be a specific point on a unit of production that does not meet requirements. Any individual unit in the inspection unit can have more than one nonconformity. If each of the inspection units are the same size or contain the same quantity of production units and the probability of observing any one of an infinite number of possible nonconformities in each inspection unit is constant, then the occurrence of nonconformities in an inspection unit follows a Poisson distribution with parameter c. There are two cases to consider: c known and c unknown.

Let X_i be the number of nonconformities in the i^{th} inspection unit. The control limits for the c-chart with a known value of c are:

$$\begin{aligned} \text{UCL} &= c + 3\sqrt{c} \\ \text{Centerline} &= c \\ \text{LCL} &= c - 3\sqrt{c}. \end{aligned} \tag{6}$$

2.3.1 The $\bar{x}$- and R-charts

To construct the combination of the $\bar{x}$- and R-charts, information about the relationship between the sample range, R, and the standard deviation from a normal distribution is needed. Let X be distributed normally with mean μ and standard deviation σ. Suppose a sample of size n is drawn from the population of X. The sample range is given by $R = x_{max} - x_{min}$. The relative range, $W = \frac{R}{\sigma}$, is a random variable with mean $\mu_W = d_2$ and standard deviation $\sigma_W = d_3$. Values of d_2 and d_3 for various sample sizes can be found in Table 1 in the Appendix.

Suppose an investigator is interested in controlling the mean value of some quality characteristic. Let random variable X be an observed value of the quality characteristic of interest from a unit sampled from an in-control process. Suppose it is known that X is distributed normally with known values of μ and σ when the process is running in control. If a sample of n independent units is taken from this population, $\bar{X}$ will be normally distributed with mean μ and standard deviation $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$. The control limits for the $\bar{x}$-chart for known μ and σ are given by:

$$\begin{aligned} \text{UCL} &= \mu + 3\tfrac{\sigma}{\sqrt{n}} \\ \text{Centerline} &= \mu \\ \text{LCL} &= \mu - 3\tfrac{\sigma}{\sqrt{n}}. \end{aligned} \tag{8}$$

The equation for W can be rewritten as $R = W\sigma$. The mean of R can be shown to be $\mu_R = d_2\sigma$ and the standard deviation can be shown to be $\sigma_R = d_3\sigma$. Using these values, the

Because the $\bar{x}$ chart is dependent upon the variability of the process being in control, it is good practice to first check if the preliminary values of R_i indicate in-control process variability. When testing with the R-chart, use rule 1 only. If the R-chart trial limits are accepted as valid using the preliminary test procedure discussed for the p-chart, perform the same test on the $\bar{x}$-chart, using any subset of the rules proposed earlier. If both sets of control limits are accepted as valid, proceed with process control analysis.

To use the $\bar{x}$- and R-charts, compute $\bar{x}_i$ and R_i for each sample taken from the process and plot these values on the corresponding chart. Use a subset of the previously discussed rules (use only rule 1 for the R-chart) to test if the process is running in control. If both charts show an out-of-control signal for the same observation number, it is suggested to search for an assignable cause for a change in variability first. Bringing the process variability under control may return the process to the in-control state on the $\bar{x}$-chart.

2.3.2 The $\bar{x}$- and s-charts

The $\bar{x}$- and R-charts work well when the sample size is constant and relatively small. For larger sample sizes, say $n > 10$, the sample range fails to account for much of the information provided by the sample when the $n - 2$ middle observations are ignored. For this reason, it is suggested that the $\bar{x}$- and s-charts be used when the sample size is greater than 10.

It is important to note that s^2, the sample variance, is an unbiased estimator for σ^2. It is not true that s is an unbiased estimator for σ. It is true that s is an unbiased estimator for $c_4\sigma$, where

$$c_4 = \left(\frac{2}{n-1}\right)^{\frac{1}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}.$$

$E(\frac{\bar{s}}{c_4}) = \sigma$. The trial control limits for the $\bar{x}$-chart are:

$$\begin{aligned} \text{UCL} &= \bar{\bar{x}} + 3\tfrac{\bar{s}}{c_4\sqrt{n}} \\ \text{Centerline} &= \bar{\bar{x}} \\ \text{LCL} &= \bar{\bar{x}} - 3\tfrac{\bar{s}}{c_4\sqrt{n}}. \end{aligned} \tag{13}$$

The trial control limits for the s-chart are:

$$\begin{aligned} \text{UCL} &= c_4\hat{\sigma} + 3\hat{\sigma}\sqrt{1-c_4^2} = \bar{s} + 3\tfrac{\bar{s}}{c_4}\sqrt{1-c_4^2} = \bar{s}(1 + \tfrac{3}{c_4}\sqrt{1-c_4^2}) = B_4\bar{s} \\ \text{Centerline} &= \bar{s} \\ \text{LCL} &= c_4\hat{\sigma} - 3\hat{\sigma}\sqrt{1-c_4^2} = \bar{s} - 3\tfrac{\bar{s}}{c_4}\sqrt{1-c_4^2} = \bar{s}(1 - \tfrac{3}{c_4}\sqrt{1-c_4^2}) = B_3\bar{s}, \end{aligned} \tag{14}$$

where B_3 and B_4 can be found in Table 1 in the Appendix. These trial control limits must be tested in the same fashion as the trial control limits for the $\bar{x}$- and R-charts were, that is, plot the s_i values on the s-chart analogously to the way the R_i values are plotted on the R-chart. Once acceptable control limits have been found for both charts, proceed with process control analysis.

2.3.3 The I-chart and MR-chart

It may be necessary, in some circumstances, to restrict the size of each sample to n=1. When this is the case, the methods for estimating process variability which have been discussed so far are not applicable. In this case, the moving range, MR, is used as an estimate of the process variability. The moving range is computed by taking the absolute value of the difference between two consecutive observations. The i^{th} moving range is given by $MR_i = |x_i - x_{i-1}|$.

the trial control limits for the $\bar{x}$-chart. If the trial control limits are satisfactory, proceed with process control analysis.

3 Cumulative Sum Control Charts

The $\bar{x}$-charts discussed in the previous section are ideal methods for monitoring process means when the magnitude of the shift in the mean required to be detected is relatively large. If the actual process shift is in the range of $.5\sigma_{\bar{x}}$ to $1\sigma_{\bar{x}}$, the $\bar{x}$-chart will be slow in detecting the shift. This is a major drawback of variables control charts. An alternative method to use when the shift in the process mean required to be detected is relatively small is the cumulative sum (cusum) procedure. The cusum procedure is also effective for detecting large shifts in the process, and its performance is comparable to Shewhart control charts in this situation (Ewan, 1963).

The basic idea behind the cusum procedure is that, if the process is in the in-control state, the difference between any particular sample average, $\bar{x}_j$, and the aim value for the process, μ_0, is expected to be zero. If i sample means are computed, the sum of the deviations between the i sample means and μ_0 is expected to be zero. This sum of deviations is called the cusum, denoted S_i, and is computed by:

$$S_i = \sum_{j=1}^{i} (\bar{x}_j - \mu_0). \tag{17}$$

For an in-control process, the value of S_i should randomly fluctuate about zero. If too many positive deviations accumulate, the value of S_i will increase continuously, indicating an increase in the process mean. If too many negative deviations accumulate, the value of

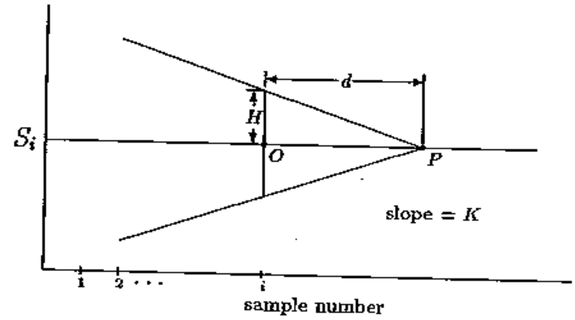

Figure 2: The V-mask

It can easily be shown that $d = \frac{H}{K}$. Thus, the construction of the V-mask is dependent upon the values H and K. H and K can be written in terms of the standard deviation of $\bar{x}$ as follows:

$$H = h\sigma_{\bar{x}} \qquad K = k\sigma_{\bar{x}}.$$

In order to obtain values of H and K, some information about an average run length (ARL) should be known.

The ARL is the average number of samples taken from a process before an out-of-control signal is detected. The in-control ARL is the average number of samples taken from an in-control process before a false out-of-control signal is detected. The in-control ARL should be chosen to be sufficiently large to reduce unnecessary adjustments to the process due to false out-of-control signals. The out-of-control ARL for a shift in the process mean from μ_0 to $\mu_1 = \mu_0 \pm \delta\sigma_{\bar{x}}$ is the average number of samples taken before a shift in the mean of $\delta\sigma_{\bar{x}}$ or greater is detected. It is desirable to detect a true shift in the process mean in as few samples as possible. Therefore, the out-of-control ARL should be satisfactorily small. Table 2 in the Appendix gives ARL's for given values of h and k.

When designing the V-mask, the first step is to specify $\Delta = \delta\sigma_{\bar{x}}$, the magnitude of the

With the value of $\hat{\sigma}_{\bar{x}}$, k can be computed by $k = \frac{K}{\hat{\sigma}_{\bar{x}}}$. Using the value of k and a specified in-control ARL, the value of h can be found from the ARL table. With this value of h, the out-of-control ARL corresponding to a shift of $\delta = \frac{\Delta}{\hat{\sigma}_{\bar{x}}}$ standard deviations should be inspected to see if it is sufficiently small. Adjustments to h and k can be made to find an acceptable balance between the in-control and out-of-control ARL's.

Another way to choose H and K is to enter the ARL table with desired values for the in-control ARL and for the out-of-control ARL for a specified value of δ, say δ_0. The columns contain ARL values for various values of δ. The column labeled $\delta = 0$ represents the in-control ARL. By following the $\delta = 0$ column and the $\delta = \delta_0$ column down until an acceptable pair of in-control and out-of-control ARL's is found, the values of h and k can be found by tracing that row over to the first two columns labeled h and k.

Once acceptable values of h and k have been found, H and K can be computed. The V-mask can be constructed as in Figure 2. The V-mask should be used in on-line process control, meaning that for each new sample taken from the process, S_i should be computed and plotted on the cusum chart. Place the V-mask with point O positioned over the most currently plotted point, making sure that the line $\overline{OP}$ is parallel to the horizontal axis. If any of the previously plotted values of S_i lie outside either of the arms of the V-mask, a shift in the process mean of magnitude $\delta\sigma_{\bar{x}}$ or greater has occurred and the process is running in the out-of-control state.

If an out-of-control process is detected, a search for an assignable cause should ensue. Once the assignable cause has been found and the proper adjustments to the process have been made, the cusum, S_i, should be reset to zero and the cusum procedure should be restarted.

consecutive samples for which $S_L(i) > 0$. If a value in either the $S_H(i)$ or $S_L(i)$ columns exceeds H, where H is computed as before, then an out-of-control signal is indicated. An investigation for an assignable cause should be carried out and the process should be adjusted accordingly. Both cusums get reset to zero and the cusum procedure is restarted once the adjustments have been made.

To estimate the new mean value of the process characteristic, use:

$$\hat{\mu} = \begin{cases} \mu_0 + K + \frac{S_H(i)}{N_H} & \text{if } S_H(i) > H \\ \mu_0 - K - \frac{S_L(i)}{N_L} & \text{if } S_L(i) > H. \end{cases} \tag{19}$$

Use the value of N_H or N_L at which the out of control signal was detected.

3.2.1 Fast Initial Response Cusum

The situation may arise where the researcher is concerned that the process may be in the out-of-control state at start-up or when the process is restarted after an adjustment has been made to the process. The standard cusum procedure discussed in the previous section may be slow in detecting a shift in the process that is present immediately upon start-up or restart. Lucas and Crosier (1982) suggested a modification to the standard cusum procedure that would decrease the response time required to detect an out-of-control signal that is present immediately upon start-up or restart. The suggested modification is to set $S(0) = S_L(0) = S_H(0)$ equal to some specified positive, nonzero constant. Lucas and Crosier suggested setting $S(0) = \frac{H}{2}$, where H is the decision interval discussed earlier.

The design of the fast initial response (FIR) cusum is similar to the design of the standard cusum. There are two differences between the two designs. First, rather than using the ARL table for a standard cusum design, one must use a table of ARL's for the FIR cusum design.

Sample	x_1	x_2	x_3	x_4	x_5
1	−30	+50	−20	+10	+30
2	0	+50	−60	−20	+30
3	−50	+10	+20	+30	+20
4	−10	−10	+30	−20	+50
5	+20	−40	+50	+20	+10
6	0	0	+40	−40	+20
7	0	0	+20	−20	−10
8	+70	−30	+30	−10	0
9	0	0	+20	−20	+10
10	+10	+20	+30	+10	+50
11	+40	0	+20	0	+20
12	+30	+20	+30	+10	+40
13	+30	−30	0	+10	+10
14	+30	−10	+50	−10	−30
15	+10	−10	+50	+40	0
16	0	0	+30	−10	0
17	+20	+20	+30	+30	−20
18	+10	−20	+50	+30	+10
19	+50	−10	+40	+20	0
20	+50	0	0	+30	+10

Figure 3: Deviations from the nominal value for holes drilled in carbon-fiber material.

of each sample was less than ten and constant from sample to sample. Because the data represent deviations from the nominal value, a mean value of $\mu_0 = 0$ was used as the target value for the cusum scheme and as the centerline for the $\bar{x}$-chart.

To set up the $\bar{x}$- and R-charts, an estimate of the process standard deviation needed to be computed. By default, SAS uses the estimate $\hat{\sigma} = \frac{\bar{R}}{d_2}$ (SAS Institute Inc.,1989), which was defined in section 2.3.1. For this example, $\bar{R} = 63.5$, $d_2 = 2.326$, and $\hat{\sigma} = 27.30009$. The $3\sigma_{\bar{x}}$ control limits for the $\bar{x}$- and R-charts can be seen in Figure 4 in the Appendix.

To analyze the control charts, one would first inspect the R-chart in Figure 4 to check that the process variability is within reasonable limits. Assuming that the control limits produced for this control chart are acceptable, one can conclude that the process variability is in control.

Investigating the $\bar{x}$-chart in Figure 4 shows that, if all seven of the decision rules proposed

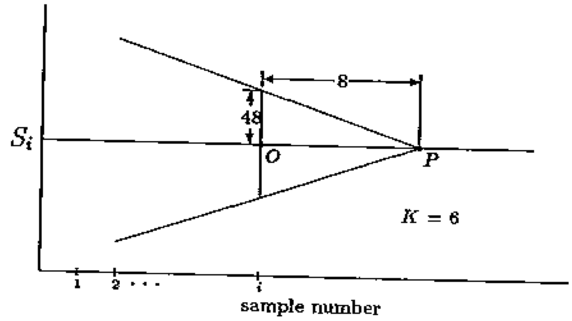

Figure 6: The V-mask with dimensions included.

the number of samples used to compute $\hat{\sigma}$. For this example, it can easily be shown that $\hat{\sigma} = 26.84091027$ and $\hat{\sigma}_{\bar{x}} = 12.0036199$.

The estimate of $\hat{\sigma}_{\bar{x}}$ was required to compute $H = h\hat{\sigma}_{\bar{x}}$ and $K = k\hat{\sigma}_{\bar{x}}$. Note that SAS uses $H = h'$ and $K = k'$. The values of H and K can be found in Figure 5 to be 48.0152154 and 6.00190193, respectively. The lead distance, d, can be shown to be 8.

The complete V-mask which would accompany this cusum scheme is illustrated in Figure 6, with its approximate dimensions included. The use of the V-mask is illustrated on the cusum chart in Figure 5 in the Appendix. The V-mask has been overlayed on the first point at which an out-of-control signal was detected. The out-of-control signal was detected at sample 15. Sample 10 was the last point to plot within the V-mask. At this point a search for an assignable cause should have been conducted. Once an assignable cause was found, the process should have been adjusted and the cusum reset to 0. None of these actions were taken in this example because the data was historical.

The tabular form of the cusum was conducted along with the cusum chart. The same parameter values were used for this tabular cusum scheme as for setting up the V-mask. The cusum table in Figure 7 in the Appendix shows the same results as the cusum chart with the V-mask. At sample 15, the upper cusum, $S_H(15)$, was 59.98478. Recall that $S_H(i)$

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Factors for constructing variables control charts

Observations in Sample, n	Chart for Averages: Factors for Control Limits			Chart for Standard Deviations: Factors for Center Line		Chart for Standard Deviations: Factors for Control Limits				Chart for Ranges: Factors for Center Line		Chart for Ranges: Factors for Control Limits				
	A	A_2	A_3	c_4	$1/c_4$	B_3	B_4	B_5	B_6	d_2	$1/d_2$	d_3	D_1	D_2	D_3	D_4
2	2.121	1.880	2.659	0.7979	1.2533	0	3.267	0	2.606	1.128	0.8865	0.853	0	3.686	0	3.267
3	1.732	1.023	1.954	0.8862	1.1284	0	2.568	0	2.276	1.693	0.5907	0.888	0	4.358	0	2.575
4	1.500	0.729	1.628	0.9213	1.0854	0	2.266	0	2.088	2.059	0.4857	0.880	0	4.698	0	2.282
5	1.342	0.577	1.427	0.9400	1.0638	0	2.089	0	1.964	2.326	0.4299	0.864	0	4.918	0	2.115
6	1.225	0.483	1.287	0.9515	1.0510	0.030	1.970	0.029	1.874	2.534	0.3946	0.848	0	5.078	0	2.004
7	1.134	0.419	1.182	0.9594	1.04230	0.118	1.882	0.113	1.806	2.704	0.3698	0.833	0.204	5.204	0.076	1.924
8	1.061	0.373	1.099	0.9650	1.0363	0.185	1.815	0.179	1.751	2.847	0.3512	0.820	0.388	5.306	0.136	1.864
9	1.000	0.337	1.032	0.9693	1.0317	0.239	1.761	0.232	1.707	2.970	0.3367	0.808	0.547	5.393	0.184	1.816
10	0.949	0.308	0.975	0.9727	1.0281	0.284	1.716	0.276	1.669	3.078	0.3249	0.797	0.687	5.469	0.223	1.777
11	0.905	0.285	0.927	0.9754	1.0252	0.321	1.679	0.313	1.637	3.173	0.3152	0.787	0.811	5.535	0.256	1.744
12	0.866	0.266	0.886	0.9776	1.0229	0.354	1.646	0.346	1.610	3.258	0.3069	0.778	0.922	5.594	0.283	1.717
13	0.832	0.249	0.850	0.9794	1.0210	0.382	1.618	0.374	1.585	3.336	0.2998	0.770	1.025	5.647	0.307	1.693
14	0.802	0.235	0.817	0.9810	1.0194	0.406	1.594	0.399	1.563	3.407	0.2935	0.763	1.118	5.696	0.328	1.672
15	0.775	0.223	0.789	0.9823	1.0180	0.428	1.572	0.421	1.544	3.472	0.2880	0.756	1.203	5.741	0.347	1.653
16	0.750	0.212	0.763	0.9835	1.0168	0.448	1.552	0.440	1.526	3.532	0.2831	0.750	1.282	5.782	0.363	1.637
17	0.728	0.203	0.739	0.9845	1.0157	0.466	1.534	0.458	1.511	3.588	0.2787	0.744	1.356	5.820	0.378	1.622
18	0.707	0.194	0.718	0.9854	1.0148	0.482	1.518	0.475	1.496	3.640	0.2747	0.739	1.424	5.856	0.391	1.608
19	0.688	0.187	0.698	0.9862	1.0140	0.497	1.503	0.490	1.483	3.689	0.2711	0.734	1.487	5.891	0.403	1.597
20	0.671	0.180	0.680	0.9869	1.0133	0.510	1.490	0.504	1.470	3.735	0.2677	0.729	1.549	5.921	0.415	1.585
21	0.655	0.173	0.663	0.9876	1.0126	0.523	1.477	0.516	1.459	3.778	0.2647	0.724	1.605	5.951	0.425	1.575
22	0.640	0.167	0.647	0.9882	1.0119	0.534	1.466	0.528	1.448	3.819	0.2618	0.720	1.659	5.979	0.434	1.566
23	0.626	0.162	0.633	0.9887	1.0114	0.545	1.455	0.539	1.438	3.858	0.2592	0.716	1.710	6.006	0.443	1.557
24	0.612	0.157	0.619	0.9892	1.0109	0.555	1.445	0.549	1.429	3.895	0.2567	0.712	1.759	6.031	0.451	1.548
25	0.600	0.153	0.606	0.9896	1.0105	0.565	1.435	0.559	1.420	3.931	0.2544	0.708	1.806	6.056	0.459	1.541

For $n > 25$

$$A = \frac{3}{\sqrt{n}}, \quad A_3 = \frac{3}{c_4\sqrt{n}}, \quad c_4 \simeq \frac{4(n-1)}{4n-3},$$

$$B_3 = 1 - \frac{3}{c_4\sqrt{2(n-1)}}, \quad B_4 = 1 + \frac{3}{c_4\sqrt{2(n-1)}},$$

$$B_5 = c_4 - \frac{3}{\sqrt{2(n-1)}}, \quad B_6 = c_4 + \frac{3}{\sqrt{2(n-1)}}.$$

Table 1: Factors for constructing variables control charts.

Source: Montgomery, Douglas C., *Introduction to Statistical Quality Control Second Edition*, John Wiley & Sons, New York, 1991, p. A-14.

Parameters		δ (shift in mean)				
h	k	0.00	0.25	0.50	0.75	1.00
2.50	0.25	13.64	11.22	7.67	5.38	4.06
4.00	0.25	38.54	24.71	13.20	8.38	6.06
6.00	0.25	125.40	50.33	20.89	12.37	8.73
8.00	0.25	368.39	83.63	28.76	16.37	11.39
10.00	0.25	1035.75	124.55	36.71	20.37	14.06
2.00	0.50	19.27	15.25	9.63	6.27	4.44
3.00	0.50	58.80	36.24	17.20	9.67	6.40
4.00	0.50	167.68	74.22	26.63	13.29	8.38
5.00	0.50	465.44	139.49	38.00	17.05	10.38
6.00	0.50	1276.56	249.26	51.34	20.90	12.37
1.50	0.75	21.28	17.22	11.01	7.00	4.77
2.25	0.75	69.85	45.97	22.04	11.63	7.13
3.00	0.75	221.40	110.95	39.31	17.34	9.68
3.75	0.75	687.85	251.56	65.58	24.16	12.37
4.50	0.75	2125.85	552.11	105.09	32.09	15.15
1.00	1.00	17.65	15.03	10.39	6.88	4.72
1.50	1.00	46.92	35.70	20.31	11.49	7.07
2.00	1.00	129.34	84.00	37.93	18.14	10.00
2.50	1.00	358.00	191.48	67.76	27.25	13.43
3.00	1.00	981.39	423.29	117.32	39.47	17.35
3.50	1.00	2670.70	917.89	199.40	55.69	21.76
0.70	1.50	33.86	28.41	18.90	11.84	7.59
1.10	1.50	92.14	71.41	40.91	22.29	12.71
1.50	1.50	274.84	191.58	91.58	42.39	21.07
1.90	1.50	881.05	536.07	208.31	80.41	34.25
2.30	1.50	2948.65	1523.15	474.09	150.96	54.47

(continued)

Table 2: Average run lengths for cusum charts.

Source: SAS Institute, Inc. *SAS/QC Software: Reference, Version 6, First Edition*, Cary,NC: SAS Institute, Inc., 1989, p. 199-200.

(continued)

Parameters		δ (shift in mean)					
h	k	1.50	2.00	2.50	3.00	4.00	5.00
2.50	0.25	2.71	2.06	1.68	1.42	1.11	1.01
4.00	0.25	3.91	2.93	2.38	2.05	1.61	1.23
6.00	0.25	5.51	4.07	3.26	2.74	2.13	1.90
8.00	0.25	7.11	5.21	4.15	3.48	2.67	2.14
10.00	0.25	8.71	6.36	5.04	4.20	3.20	2.65
2.00	0.50	3.74	1.99	1.58	1.32	1.07	1.01
3.00	0.50	3.75	2.68	2.12	1.77	1.31	1.07
4.00	0.50	4.75	3.34	2.62	2.19	1.71	1.31
5.00	0.50	5.75	4.01	3.11	2.57	2.01	1.69
6.00	0.50	6.75	4.68	3.62	2.98	2.24	1.95
1.50	0.75	2.73	1.90	1.48	1.24	1.04	1.00
2.25	0.75	3.73	2.51	1.91	1.56	1.16	1.02
3.00	0.75	4.73	3.12	2.36	1.93	1.41	1.11
3.75	0.75	5.73	3.71	2.79	2.27	1.72	1.31
4.50	0.75	6.73	4.31	3.21	2.59	1.97	1.60
1.00	1.00	2.63	1.78	1.38	1.17	1.02	1.00
1.50	1.00	3.50	2.24	1.66	1.34	1.07	1.01
2.00	1.00	4.45	2.74	1.99	1.58	1.16	1.02
2.50	1.00	5.42	3.25	2.34	1.85	1.31	1.07
3.00	1.00	6.40	3.75	2.68	2.12	1.52	1.16
3.50	1.00	7.39	4.25	3.01	2.37	1.73	1.31
0.70	1.50	3.66	2.18	1.55	1.25	1.04	1.00
1.10	1.50	5.17	2.80	1.86	1.43	1.08	1.01
1.50	1.50	7.09	3.50	2.24	1.66	1.16	1.02
1.90	1.50	9.38	4.26	2.64	1.92	1.29	1.05
2.30	1.50	12.00	5.03	3.04	2.20	1.45	1.12

Table 2: Average run lengths for cusum charts.

Source: SAS Institute, Inc. *SAS/QC Software: Reference, Version 6, First Edition*, Cary,NC: SAS Institute, Inc., 1989, p. 199-200.

Average Run Lengths for One-Sided CUSUM Scheme: $S_0 = h/2$

PARAMETERS			DISPLACEMENT OF CURRENT MEAN (MULTIPLE OF TRUE VALUE OF SIGMA)										
H	K	S(O)	.000	.250	.500	.750	1.00	1.50	2.00	2.50	3.00	4.00	5.00
2.50	.25	1.25	22.87	10.32	5.668	3.664	2.667	1.760	1.368	1.169	1.068	1.006	1.000
4.00	.25	2.00	66.57	20.25	8.991	5.291	3.700	2.354	1.770	1.441	1.233	1.040	1.003
6.00	.25	3.00	225.4	38.76	13.48	7.382	5.054	3.169	2.372	1.932	1.638	1.227	1.040
8.00	.25	4.00	684.3	63.25	17.86	9.416	6.390	3.967	2.940	2.383	2.046	1.605	1.227
10.00	.25	5.00	1972.	93.75	22.13	11.43	7.724	4.767	3.509	2.815	2.381	1.933	1.599
2.00	.50	1.00	34.40	15.19	7.785	4.626	3.126	1.873	1.395	1.174	1.069	1.006	1.000
3.00	.50	1.50	108.0	33.39	13.25	6.755	4.208	2.353	1.680	1.348	1.165	1.023	1.001
4.00	.50	2.00	316.4	66.57	20.25	8.991	5.291	2.862	2.014	1.586	1.325	1.067	1.006
5.00	.50	2.50	895.9	124.9	28.76	11.24	6.348	3.372	2.362	1.856	1.540	1.159	1.023
6.00	.50	3.00	2492.	225.4	38.76	13.48	7.382	3.875	2.703	2.125	1.774	1.311	1.067
1.50	.75	.75	39.44	18.70	9.747	5.658	3.659	2.015	1.431	1.182	1.070	1.006	1.000
2.25	.75	1.12	131.9	46.22	18.72	9.004	5.140	2.499	1.670	1.314	1.138	1.017	1.001
3.00	.75	1.50	426.6	108.0	33.39	13.25	6.755	3.010	1.950	1.489	1.243	1.040	1.003
3.75	.75	1.87	1345.	242.8	56.40	18.36	8.428	3.528	2.251	1.696	1.386	1.085	1.009
4.50	.75	2.25	4193.	534.0	91.74	24.32	10.12	4.043	2.560	1.922	1.559	1.160	1.023
1.00	1.00	.50	33.54	17.78	10.04	6.084	3.982	2.135	1.466	1.190	1.071	1.006	1.000
1.50	1.00	.75	89.74	39.44	18.70	9.747	5.658	2.608	1.657	1.282	1.115	1.012	1.001
2.00	1.00	1.00	250.2	88.45	34.40	15.19	7.785	3.126	1.873	1.395	1.174	1.023	1.001
2.50	1.00	1.25	699.9	195.9	61.68	22.87	10.32	3.664	2.107	1.529	1.252	1.040	1.003
3.00	1.00	1.50	1934.	426.6	108.0	33.39	13.25	4.208	2.353	1.680	1.348	1.067	1.006
3.50	1.00	1.75	5292.	918.6	185.9	47.59	16.56	4.752	2.606	1.843	1.460	1.107	1.012
.70	1.50	.35	66.43	34.92	19.31	11.27	6.966	3.204	1.887	1.367	1.146	1.016	1.001
1.10	1.50	.55	181.2	83.78	40.61	20.79	11.36	4.279	2.225	1.502	1.207	1.026	1.002
1.50	1.50	.75	542.8	216.1	89.74	39.44	18.70	5.658	2.608	1.657	1.282	1.041	1.003
1.90	1.50	.95	1748.	585.2	203.5	75.30	30.51	7.325	3.020	1.828	1.371	1.062	1.005
2.30	1.50	1.15	5871.	1619.	464.3	142.8	48.99	9.260	3.447	2.012	1.473	1.091	1.009

PARAMETERS:

H - DECISION INTERVAL FOR THE CUSUM SCHEME

K - REFERENCE VALUE

S(O) - HEADSTART VALUE

Average Run Lengths for Two-Sided CUSUM Scheme: $S_0 = h/2$

PARAMETERS			DISPLACEMENT OF CURRENT MEAN (MULTIPLE OF TRUE VALUE OF SIGMA)										
H	K	S(O)	.000	.250	.500	.750	1.00	1.50	2.00	2.50	3.00	4.00	5.00
2.50	.25	1.25	9.236	7.553	5.083	3.515	2.624	1.755	1.367	1.169	1.068	1.006	1.000
4.00	.25	2.00	28.03	17.36	8.682	5.240	3.690	2.354	1.770	1.441	1.233	1.040	1.003
6.00	.25	3.00	100.0	36.77	13.38	7.372	5.053	3.169	2.372	1.932	1.638	1.227	1.040
8.00	.25	4.00	315.9	62.16	17.83	9.414	6.390	3.967	2.940	2.383	2.046	1.605	1.227
10.00	.25	5.00	936.2	93.21	22.12	11.43	7.724	4.767	3.509	2.815	2.381	1.933	1.599
2.00	.50	1.00	15.13	11.79	7.178	4.487	3.086	1.869	1.395	1.174	1.069	1.006	1.000
3.00	.50	1.50	49.19	29.33	12.88	6.694	4.195	2.352	1.680	1.348	1.165	1.023	1.001
4.00	.50	2.00	148.7	62.70	20.06	8.968	5.287	2.862	2.014	1.586	1.325	1.067	1.006
5.00	.50	2.50	430.4	121.7	28.67	11.24	6.347	3.372	2.362	1.856	1.540	1.159	1.023
6.00	.50	3.00	1215.	222.9	38.71	13.48	7.382	3.875	2.703	2.125	1.774	1.311	1.067
1.50	.75	.75	18.15	14.51	8.997	5.498	3.617	2.010	1.430	1.182	1.070	1.006	1.000
2.25	.75	1.12	62.08	40.02	18.16	8.918	5.121	2.497	1.670	1.314	1.138	1.017	1.001
3.00	.75	1.50	205.2	100.3	33.03	13.21	6.747	3.009	1.949	1.489	1.243	1.040	1.003
3.75	.75	1.87	656.9	234.1	56.19	18.34	8.425	3.528	2.251	1.696	1.386	1.085	1.009
4.50	.75	2.25	2068.	524.6	91.62	24.31	10.12	4.042	2.560	1.922	1.559	1.160	1.023
1.00	1.00	.50	15.89	13.44	9.118	5.882	3.931	2.131	1.466	1.190	1.071	1.006	1.000
1.50	1.00	.75	42.82	32.22	17.75	9.597	5.627	2.606	1.657	1.282	1.115	1.012	1.001
2.00	1.00	1.00	120.8	77.40	33.57	15.10	7.767	3.125	1.873	1.395	1.174	1.023	1.001
2.50	1.00	1.25	341.9	180.3	61.02	22.81	10.31	3.663	2.107	1.529	1.252	1.040	1.003
3.00	1.00	1.50	952.6	405.5	107.5	33.35	13.25	4.208	2.353	1.680	1.348	1.067	1.006
3.50	1.00	1.75	2621.	890.8	185.5	47.57	16.56	4.752	2.606	1.843	1.460	1.107	1.012
.70	1.50	.35	32.57	27.22	17.90	11.02	6.917	3.202	1.886	1.367	1.146	1.016	1.001
1.10	1.50	.55	89.02	68.64	38.71	20.55	11.33	4.278	2.225	1.502	1.207	1.026	1.002
1.50	1.50	.75	267.9	185.8	87.31	39.22	18.67	5.657	2.608	1.657	1.282	1.041	1.003
1.90	1.50	.95	866.4	524.6	200.6	75.13	30.49	7.325	3.020	1.828	1.371	1.062	1.005
2.30	1.50	1.15	2920.	1502.	461.1	142.7	48.98	9.260	3.447	2.012	1.473	1.091	1.009

PARAMETERS:

H - DECISION INTERVAL FOR THE CUSUM SCHEME

K - REFERENCE VALUE

S(O) - HEADSTART VALUE

Table 3: Average run lengths for cusum charts with FIR option.

Source: Lucas, James M., and Crosier, Ronald B., "Fast Initial Response for CUSUM Quality Control Schemes: Give Your CUSUM A Head Start," *Technometrics*, 24, 199-205,1982.

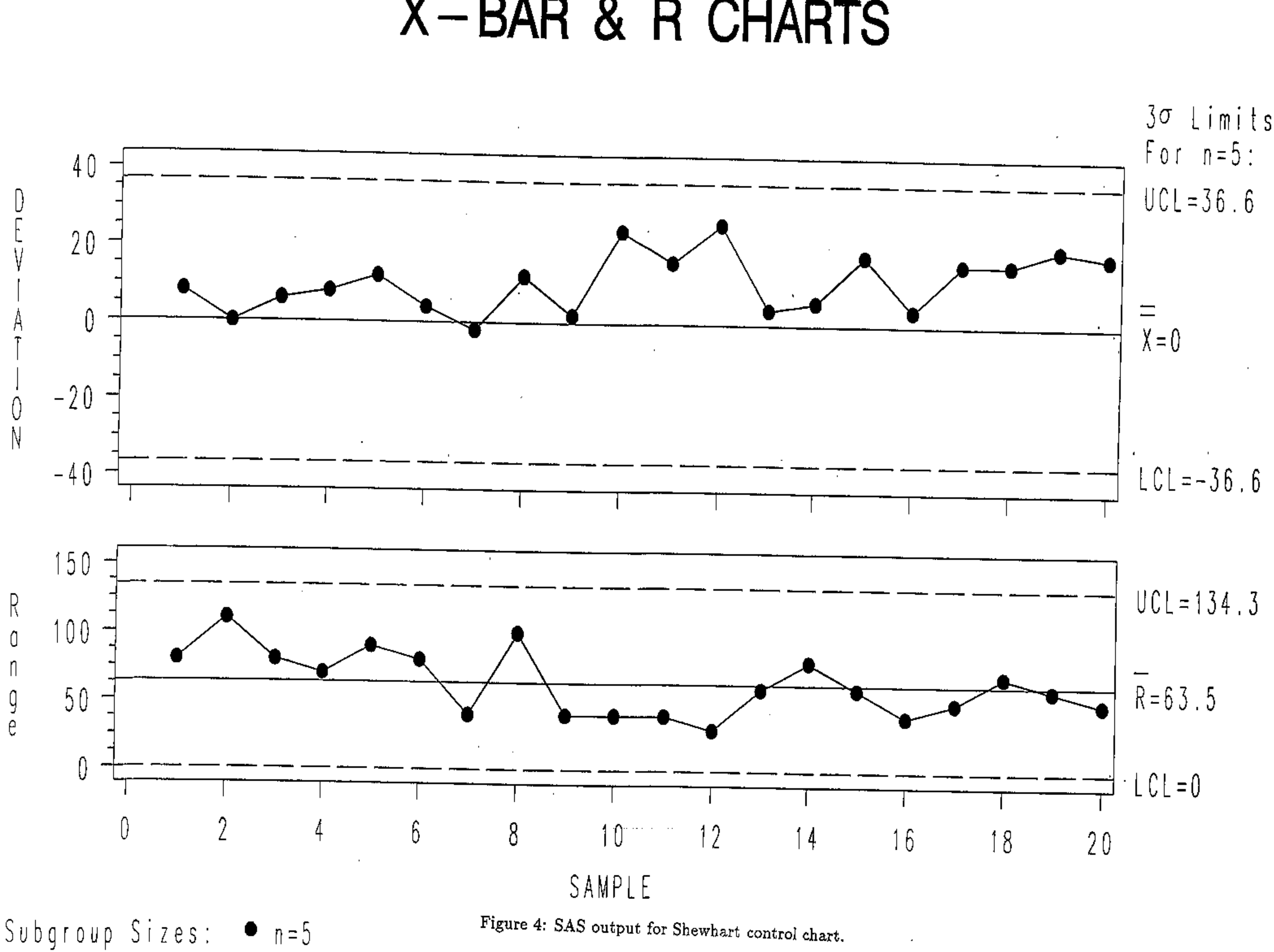

Figure 4: SAS output for Shewhart control chart.

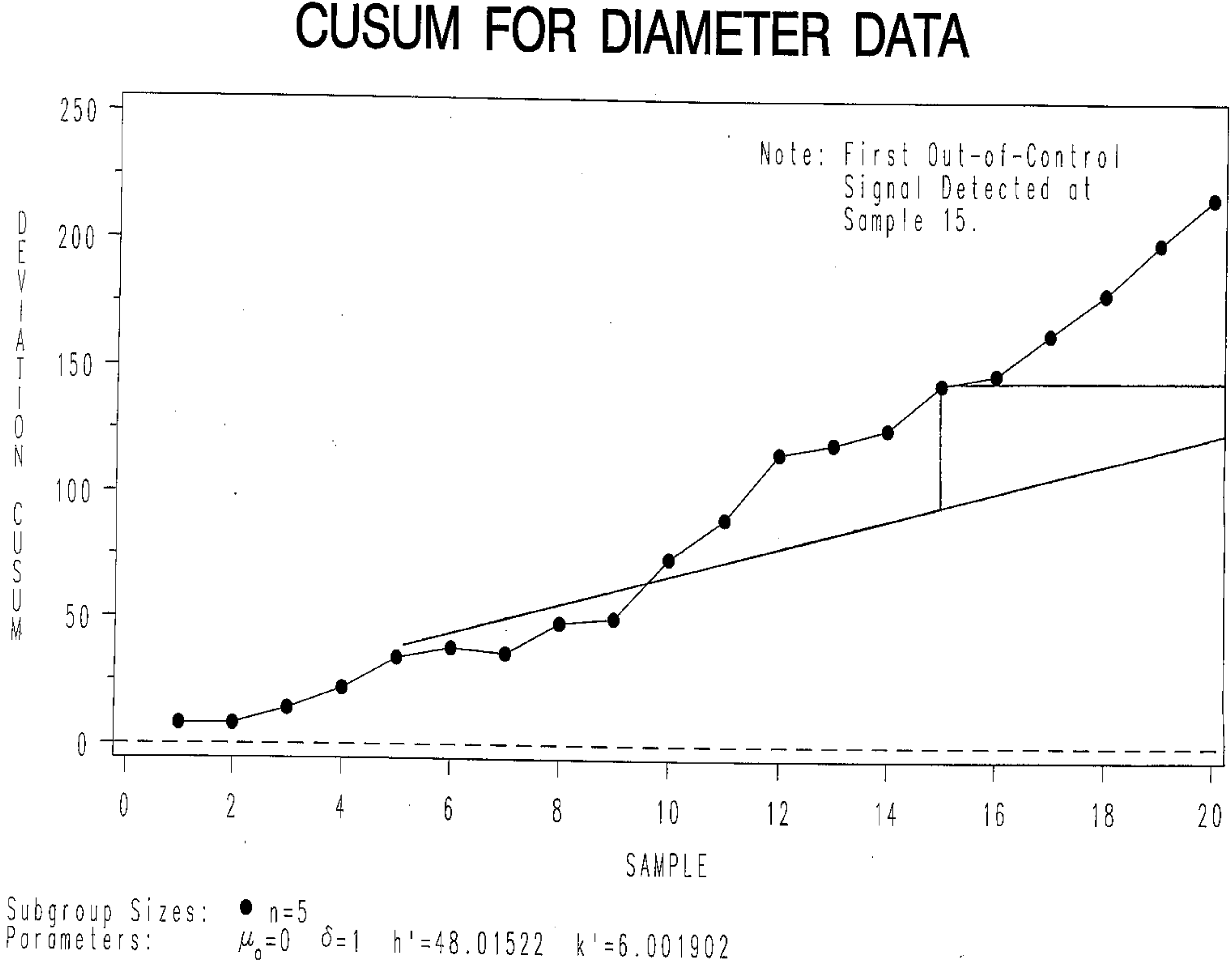

Figure 5: SAS outputfor cusum chart.

CUSUM FOR DIAMETER DATA

Summary of Cumulative Sum Control Chart Analysis

Process Variable:	DEV	(DEVIATION CUSUM)
Subgroup Variable:	N	(SAMPLE)

Mu0 (Target Mean)	0
Std Deviation	26.8413214
Delta	1
Scheme	Two-Sided
Limitn	5
h	4
h' (Data Units)	48.0152154
k	0.5
k' (Data Units)	6.00190193
ARL(Delta)	8.38313041
ARL(0)	167.683749

CUSUM FOR DIAMETER DATA

N	Subgroup Sample Size	Subgroup Mean for DEV	Subgroup Std Dev for DEV	Lower Cumulative Sum	Number of Consecutive Lower Sums > 0	Upper Cumulative Sum	Number of Consecutive Upper Sums > 0
1	5	8.000000	33.466401	0	0	1.99810	1
2	5	0.000000	43.011626	0	0	0.00000	0
3	5	6.000000	32.093613	0	0	0.00000	0
4	5	8.000000	30.331502	0	0	1.99810	1
5	5	12.000000	32.710854	0	0	7.99620	2
6	5	4.000000	29.664794	0	0	5.99429	3
7	5	-2.000000	14.832397	0	0	0.00000	0
8	5	12.000000	38.987177	0	0	5.99810	1
9	5	2.000000	14.832397	0	0	1.99620	2
10	5	24.000000	16.733201	0	0	19.99429	3
11	5	16.000000	16.733201	0	0	29.99239	4
12	5	26.000000	11.401754	0	0	49.99049	5
13	5	4.000000	21.908902	0	0	47.98859	6
14	5	6.000000	32.863353	0	0	47.98669	7
15	5	18.000000	25.884358	0	0	59.98478	8
16	5	4.000000	15.165751	0	0	57.98288	9
17	5	16.000000	20.736441	0	0	67.98098	10
18	5	16.000000	26.076810	0	0	77.97908	11
19	5	20.000000	25.495098	0	0	91.97718	12
20	5	18.000000	21.679483	0	0	103.97527	13

Figure 7: SAS output for cusum table.