Fisher's Exact Test Versus The χ^2 Approximation–for the 2×2 table

Eric J. Lloyd

August 8, 1994

1 Introduction

A large proportion, maybe even a majority, of the variables that we use in real life are categorical in nature. When I assess my day's productivity at the end of the day, I don't assign a quantitative value to the day's different tasks. Rather I say this task was "good", "average", or "poor". The same can be said about empirical studies in all types of disciplines. Categorical variables are to be found in the most rigid sciences, like physics or biology, to the more trivial, such as the flavor of bubble gum. The analysis of these categorical variables falls under a large and extensive field of study called nonparametric statistics.

Even in the hands of an experienced statistician these nonparametric data methods can be quite sophisticated. A nonparametric procedure typically considers all possible outcomes and then assigns a likelihood to the observed data based on all other possible outcomes, just like one might calculate the likelihood of a poker hand. The functioning researcher wants his/her nonparametric test to behave like the classical parametric test. Typically, this is reflected in p-values (the probability of the data given some hypothesis of the nature of the data) and confidence intervals.

From the inception of modern statistical methods the p-value has meant do-or-die. For instance, a single p-value can be the criterion for deciding whether or not a paper is published, what research is funded, which public health policy will be enforced, and even whether current judicial policy is ethical and prudent.

The main pitfall of tabled data, the most common nonparametric method, has been the generation of a p-value which correctly represents the truth of the phenomena of study. As nonparametric studies grow (such as in sample size or number of variables) the sampling distribution, i.e., the distribution from which the p-values are calculated, increases in its complexity. In the past, before the advances in computing were realized, a researcher would follow various (and often conflicting) rules of thumb which assured the researcher that large sample theory approximations were reliably accurate. That is, asymptotic theory was valid only for data which wasn't small, sparse, skewed, or heavily tied. These limitations place great limits on the practicing scientist because data is small, sparse, skewed, and heavily tied. R. A. Fisher states[3]:

The traditional machinery of statistical processes is wholly unsuited to the needs of practical research. Not only does it take a cannon to shoot a sparrow, but is misses the sparrow! The elaborate mechanism built on the theory of infinitely large samples is not accurate enough for simple laboratory data. Only by systematically tackling small problems on their merits does it seem possible to apply accurate test to practical data.

The rapid growth in computing has caused equally great advances in correctly reported nonparametric statistics. In particular, we are now able to generate exact p-values for nonparametric tests that were previously unobtainable or very lengthy in computation. Throughout this report I will be using a relatively new statistical package called StatXact, a pioneering software package in exact nonparametric methods.

Although exact probabilities are becoming more readily obtainable, some asymptotic theory will still be required in the analysis of categorical data. Therefore, the goal of this paper is to discuss and study several large sample rules of thumb for tabled data, and to comment on the feasibility of large sample theory rules of thumb when applied to the analysis of the popular two way table.

2 What is StatXact?

StatXact copy written by Cytel Software Corporation, 675 Massachusetts Avenue, Cambridge, MA 02139, is a new wonder in the world of statistical computing. Recent advances in computing power and new algorithms in computing permutations have led to the development of a package like StatXact. StatXact uses both permutation algorithms and Monte Carlo methods to derive the p-values of computationally difficult data sets.

StatXact will obtain a p-value in one of three ways:

- StatXact will first attempt to calculate the exact probability associated with a test.
- If unable to calculate the exact probability, because the data set is to large, StatXact will attempt a Monte Carlo simulation.

- Lastly, if StatXact is unable to calculate either of the previous two because of large data size, it will generate the asymptotic approximation.

Before continuing further, a few comments should be made regarding this idea of a permutational p-value and what StatXact does when computing one. A permutational p-value is derived by "constructing a reference set of all possible outcomes in which the exact null probability of each outcome is known. The exact p-value is then the sum of exact probabilities of those outcomes in the reference set that are at least as extreme as the one observed. [4]" In other words, StatXact actually determines the number of ways the data could be arranged, given certain conditions are held constant, and then counts the number of ways the data could have been arranged as extreme as, or more than, the actual realization.

The following example taken from the StatXact manual [[4], page 1-4] is a poignant illustration of potential discrepancies in p-values for permutational versus large sample methods.

Assume we have the following two-way table, which consists of 9 categories under variable 1 and 3 categories under variable 2. As the reader can note there are many cells with zero values, something not uncommon to clinical data.

	Variable 1								
Variable 2	0	7	0	0	0	0	0	1	1
	1	1	1	1	1	1	1	0	0
	0	8	0	0	0	0	0	0	0

Table 1: Hypothetical 2-way table taken

If we apply a Pearson's χ^2 test to this table we would obtain a test statistic of $\chi^2 = 22.27$ which has a corresponding p-value of 0.1342. An exact test p-value is 0.0013. The exact test indicates that there is an association between the two variables while the large sample theory test, the Pearson's χ^2 test, does not indicate any association.

3 A Quick Discussion of the Two–Way Table

The two–way table is a popular way of displaying and evaluating categorical data. Table 1 in the previous section is an illustration of a two–way table. It is called a two–way table because it defines the cross-classification of two variables. For readability one variable is placed along the rows and the other is placed along the columns. Therefore, the two–way table is often called a $r \times c$ table, where r is the number of rows or the number of categories for the row variable, and c is likewise the number columns or number of categories for the other variable. Where a row and column cross is what is called a cell, and the number found in any cell is the count of the members in the sample exhibiting that row and column characteristic, exclusively.

The two–way table is useful because a large amount of research is concerned with determining if two variables are related. Insurance companies are interested in knowing if gender and claims are related. Public health has long been interested in the association of cigarettes and lung cancer. If two variables are independent (no relationship) then the distribution of one variable in no way depends on the distribution of the other [2]. Table 2 is an example of a two–way table, showing the relationship of night time smoking with lung cancer for 56 subjects. The analysis of this 2×2, or four-fold table, for potential association is carried out in the next section.

	Night time smoking		
Lung cancer	Yes	No	Total
Yes	20	16	36
No	6	14	20
Total	26	30	56

Table 2: Data taken from Daniel [2], page 186

4 Two Popular Ways to Analyze the Two–Way Table

1. Fisher Exact Test

   The Fisher Exact Test tests the independence of two variables with multiple categories in each. It is an exact test because it calculates the likelihood of the data given that the row and column totals are set prior to the data collection. Fisher originally proposed the procedure for use only with 2 by 2 tables, but it has been expanded to higher dimensions.

n_{11}	n_{12}	$n_{1.}$
n_{21}	n_{22}	$n_{2.}$
$n_{.1}$	$n_{.2}$	$n_{..}$

Table 3: Standard setup of a 2×2 table

The following is an outline of how the test is performed according to the Daniel text *Applied Nonparametric Statistics* [2].

- Assumptions:
  - (A) The data consists of a size $n_{1.}$ sample from population 1, and a size $n_{2.}$ sample from population 2, implying the marginal totals are fixed.
  - (B) The samples are random and independent.
  - (C) Each observation can be classified as one of two mutually exclusive types.
- Possible Null and Alternative Hypothese:
  - (A) (Two-sided)
    H_0 : The proportion with characteristic of interest is the same in both populations, $p_1 = p_2$.
    H_1 : The proportion with characteristic of interest is different for both populaitons, $p_1 \neq p_2$

(B) (One-sided)
H_0 : The proportion in population 1 is less than or equal to the proportion in population 2, $p_1 \leq p_2$.
H_1 : The proportion in populaiton 1 greater than the proportion in populaiton 2, $p_1 > p_2$

- Decision Rule: Fixed marginal totals allow the use of the hypergeometric distribution. If we let $n_{1.}$ be the total for row 1, $n_{2.}$ be the total for row 2, $n_{.1}$ be the total for column 1, and $n_{.2}$ be the total for column 2, then the hypergeometric probability for observing a particular 2×2 table under the null hypothesis is

$$Pr(n_{11}, n_{12}, n_{21}, n_{22}) = \frac{n_{1.}!n_{2.}!n_{.1}!n_{.2}!}{n_{..}!n_{11}!n_{12}!n_{21}!n_{22}!} \tag{1}$$

If the sum of all hypergeometric probabilities corresponding to cases as or more extreme than the observed data (the test's p–value) is smaller than the α level, which is usually 0.01, 0.05 or 0.1, then the null hypothesis is rejected.

Fisher Exact Test as Calculated by Other Software Packages

- **SPSS-X** Will not do exact test when $n > 20$. When $n > 20$, SPSS-X runs the analysis as a χ^2 test.
- **SAS** For $r \times c$ tables, SAS uses the Mehta and Patel algorithm, the same algorithm developed by the creators of StatXact.
- **StatXact** For $r \times c$ tables, StatXact uses the Mehta and Patel algorithm which orders all possible tables, given the fixed marginal totals, and counts the number of tables considered as extreme as the the one the data produced.

2. Chi–square test (Independence)

Another method to test association of two qualitative variables is the Pearson's χ^2 test. This method is based on the idea that if there is independence then all observations are purely random. Therefore, if there are large differences in the theoretical expected cell counts with the observed counts, then one might conclude an association.

The following is an outline of how the test is performed according to Daniel[2].

- Assumptions:
  (A) Data consists of a simple random sample of size n from some population of interest.
  (B) The data may be cross–classified using two categorical criteria. Thus, each observation belongs to exactly one category of each criterion. The criteria are the variables of interest in a given situation.
  (C) The variables may be categorical or quantitative, with measurements that are capable of being classified into mutually exclusive numerical categories.
- Null and Alternative Hypothese
  - H_0 : The two criteria of classification (variables) are independent.
  - H_1 : There exist some relationship between the two criteria classification and they are not independent.
- Test Statistic: Compares observed results with expected results under the assumption that the null hypothesis is true.
  The i^{th} row and j^{th} cell expectation is defined as:

$$E_{ij} = n\left(\frac{n_{i.}}{n}\right)\left(\frac{n_{.j}}{n}\right) \qquad (2)$$

  The test statistic is defined as:

$$X^2 = \sum_{i=1}^{r}\sum_{j=1}^{c}\left[\frac{(O_{ij} - E_{ij})^2}{E_{ij}}\right] \dot{\sim} \chi^2((r-1)(c-1)) \qquad (3)$$

  Test conclusion is based on a χ^2 distibution based p-value of the test statistic.

3. Exact versus Approximated test

The names alone imply that there are some clear advantages to the exact test. The best property of using the exact procedure over the χ^2

procedure is the exact procedure preserves the integrity of the type I error. According to the StatXact manual [4] the exact test guarantees to control the type I error at the chosen α level. The same guarantee isn't to be had with large sample theory, or the χ^2 approximation. Both procedures give discrete p-values, therefore one can rarely achieve a true α level test. Because the exact test preserves the error rate in the long run it is also called a *conservative* test [4]. Much of the conservativeness is attributed to the reference set having fixed rows and column counts.

The only advantage that the χ^2 test has over the exact procedure is the computation is always possible. When sample sizes become large, or when there are many rows or columns, the exact test becomes in feasible to calculate due to the computational time required. Since the χ^2 test is always obtainable, it is a very common procedure, included in most software packages. Because of its , easy accessibility, it is reasonable to assume that the χ^2 test is often overused, especially in cases where it is reasonable to perform the exact test and the results of the two tests would cause contrasting conclusions.

For the data presented in Table 2(page 4) the test of association could be carried out in either of these two methods. For the exact procedure the association with night time smoking to lung cancer has a p-value of 0.0947 which is significant at the $\alpha = 0.1$ level but not at an $\alpha = 0.05$ level.

For the χ^2 test performed on the same data, the p–value for the association test is 0.0662. This is also significant at the $\alpha = 0.1$ level but not at an $\alpha = 0.05$ level. However, the approximated p–value underestimated the true p–value.

5 Proposed rules of thumb to be applied to χ^2 Contingency Tables

The question *When is a χ^2 test appropriate?*, is a question not unanimously agreed upon by statisticians. The following is a brief list of proposed *Rules of Thumb* offered by different members of the statistical community:

- Don't use χ^2 test if sample size is less than 20 [6].

- When $20 \leq N \leq 40$ the use of χ^2 approximation is reliable if all expected frequencies are ≥ 5 [6].
- When $N > 40$ use χ^2 corrected for continuity [6]
- When $N > 40$ use χ^2 if the smallest cell expected frequency is greater than or equal to 1 [2].
- May use χ^2 when the minimum for an expected cell value should be as high as 10 [2].
- For a contingency table with more than 1 degree of freedom, Cochran recommends that a minimum expected frequency as low as 1 be allowed if no more than 20% of the cells have expected frequencies of less than 5 [1].
- If χ^2 has less than 30 degrees of freedom and the minimum expected frequency is 2 or more, Cochran states that the use of the ordinary χ^2 table is usually adequate [1].
- Roscoe and Byars suggest: If data is uniform, χ^2 is acceptable for expected cells as low as 2 when testing at $\alpha = 0.05$ and as low as 4 when testing at $\alpha = 0.01$. If the data is slightly nonuniform, χ^2 is acceptable for expected cells as low as 4 when testing at $\alpha = 0.05$ and as low as 6 when testing at $\alpha = 0.01$ [5].

6 Investigating rules of thumb

The goal of the investigative part of this paper is to explore the applicability and truthfulness of some of the aforementioned rules of thumb. The two rules of thumb of most interest were the ones which claimed (1) don't use χ^2 test if sample size is less than 20, and (2) when $20 \leq N \leq 40$ the use of χ^2 approximation is reliable if all expected frequencies are 5 or more. Before we continue into a discussion of the experimental investigations, it is necessary to define a few terms and the layout of the tables which accompany the discussion.

A *significance level* is what is often referred to as the α level or the decision rule. The most common decision rules are to reject a null hypothesis if the p-value associated with the data is less than or equal to 0.01, 0.05, or 0.1. For the purposes of this paper all three are considered, and the primary statistical interest is whether or not both the exact p-values and the χ^2 p-values indicate the same decision.

A *rare event* is related to data with a statistically significant p-value. In the course of this paper we are examining tables conditioned on certain marginal totals, therefore a *rare table* is one where exactly one or both p-values are below one of the three decision rules.

Finally, results of the investigation are summarized in various tables, beginning with table 4 (page 11). Each table is subdivided with *cases*, where a *case* is a set of tables with the same sample size, marginal totals, and expected cell frequencies. The smallest expected cell count (see equation 2) is included for each *case*. Each table is identified by the column denoted by *Diagonals*, representing the diagonal elements for that table.

6.1 Investigating rule # 1

The first rule of thumb to be considered is *don't use χ^2 test if sample size is less than 20*[6]. Only the Fisher exact test should be performed. This appears to be a widely accepted rule. For example, the SAS statistical package will only perform an $r \times c$ table as a Fisher exact test when sample size is less than or equal to 20. A second indication of it's acceptance is that most tables for the Fisher exact test (found in statistics books) only hold for sample size of 20 or smaller.

This rule of thumb was investigated by generating all possible and unique 2×2 tables for given sample sizes. Both 2 tailed p-values were calculated for each table, the exact and the χ^2 p-value. For any given sample size N there are x number of *cases*. If this rule of thumb is reliable, then we would expect to see differences in conclusions based on exact versus approximated p-values for a table corresponding to sample size smaller than 20. Two samples were considered, first a sample of size 8, and second, a sample size of 19.

For a sample size of 8 there are 10 unique tables with 4 different cases, the most trivial table (8,0;0,0) being omitted. The results of these tables are summarized in table 4 on page 11. There are two interesting observations to be made from this sample. First, there are only 4 tables with a significant p-value (using the most liberal significance level of 0.1) implying that for a sample too small, it is unlikely to find any significant association. Second,

Sample size = 8					
Case 1: E[smallest cell] = 2.0			Case 2: E[smallest cell] = 1.125		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
4:4	0.0286	0.0067	5:3	0.0179	0.0080
3:3	0.4857	0.1789	4:2	0.4643	0.2032
2:2	1.0000	0.6669	3:1	1.0000	0.6296
Case 3: E[smallest cell] = 0.5			Case 4: E[smallest cell] = 0.125		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
6:2	0.0357	0.0136	7:1	0.1250	0.0307
5:1	0.4643	0.2888			
4:0	1.0000	0.4100			

Table 4: All unique tables for sample size of 8

all four cases, excluding case 3, had one conflicting decision between the two p-values, applying one of the three most common significance levels ($\alpha = 0.1, 0.05, 0.01$), suggesting that of the four rare tables 75% had a potentially incorrect decision.

The next sample investigated was a sample of 19 observations. This sample yielded 29 tables (not the only unique tables, just those with p-values of interest). This sample is summarized in table 5 on page 12. With this sample there were 21 rare tables with 4 rare tables having a conflicting decision rule, or 19% being falsely classified. An impressive improvement over the size 8 sample.

6.2 Investigating rule # 2

The next rule of thumb investigated states that when $20 \leq N \leq 40$ the use of χ^2 approximation is reliable if all expected frequencies are 5 or more [6]. Investigating this idea, sample sizes of 20, 25, 30, 35, and 40 were examined in the same way as the previous two samples were evaluated. However the operative part of this rule is the condition restricting that all expected frequencies be greater than or equal to 5.

The Reader may note that a sample size of 20 is the smallest possible

Sample size = 19					
Case 1: E[smallest cell] = 4.3			Case 2: E[smallest cell] = 3.4		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
10:9	0.0000	0.0000	11:8	0.0000	0.0000
9:8	0.0011	0.0006	10:7	0.0011	0.0006
8:7	0.0230	0.0118	9:6	0.0237	0.0133
Case 3: E[smallest cell] = 2.6			Case 4: E[smallest cell] = 1.9		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
12:7	0.0000	0.0000	13:6	0.0000	0.0000
11:6	0.0017	0.0007	12:5	0.0029	0.0010
10:5	0.0449	0.0170	11:4	0.0460	0.0250
Case 5: E[smallest cell] = 1.3			Case 6: E[smallest cell] = 0.8		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
14:5	0.0001	0.0000	15:4	0.0003	0.0000
13:4	0.0061	0.0015	14:3	0.0157	0.0029
12:3	0.0844	0.0463			
Case 7: E[smallest cell] = 0.5			Case 8: E[smallest cell] = 0.2		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
16:3	0.0010	0.0000	17:2	0.0058	0.0000
15:2	0.0506	0.0085	16:1	0.2047	0.0545
14:1	0.4221	0.3638			
Case 9: E[smallest cell] = 0.05					
Diagonals	Exact	Asymptotic			
18:1	0.0526	0.0000			

Table 5: Tables of interest for samples of size 19

Sample size = 20		
Case 1: E[smallest cell] = 5		
Diagonals	Exact	Asymptotic
10:10	0.0000	0.0000
9:9	0.0011	0.0003
8:8	0.0230	0.0073
7:7	0.1789	0.0736

Table 6: Tables where smallest $E[cell] \geq 5$

sample in the 2×2 table case where all expected cell frequencies can be greater or equal to 5. In fact, there is only one case in the $N = 20$ where all cells have expected counts of 5 or greater. These results (found in table 6 on page 13), show that only one case exists for this sample size conditioned on expected cell count of 5 or greater. For the four tables found to be rare 2 give conflicting decisions, or an astonishing 50% being falsely classified.

A remarkable finding is in table 14, which contains the results of other tables of size 20 which fail the condition of all cells having expected counts of 5 or higher. For these samples, 25 tables were classified as rare with 10 incorrectly classified, for a failure of 40%.

For a sample size of 25 the results are nearly the same. Only one case exists where all cells have expected count of 5 or greater, results found in table 7, page 14. For this set there are 5 rare tables with 2 conflicting tables, for a failure of 40%. If we were to look at the tables not following the count condition we have 37 rare tables with 27% failure.

The remainder of the samples 30, 35, 40 are summarized resfectively in Tables 10, 11, and 12 of section 10 *Appendix–References Tables.*

An additional sample of size 45 was investigated to see the limits of this rule of thumb. Both sets were evaluated, those satisfying the condition and those not satisfying the condition. These results are found in Table 13 through Table 19 in the Appendix.

A detailed summary of the sample 20, 25, 30, 35, 40, and 45 by decision rules ($\alpha = 0.01, 0.05, 0.1$) for the constrained and the non-constrained cases, are found in Tables 8 and 9(page 14).

Sample size = 25		
Case 1: E[smallest cell] = 5.75		
Diagonals	Exact	Asymptotic
13:12	0.0000	0.0000
12:11	0.0000	0.0000
11:10	0.0012	0.0007
10:9	0.0169	0.0094
9:8	0.1152	0.0727

Table 7: Tables of size 25 where $E[count] \geq 5$

	Constraint					
	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.1$	
Sample size	# Rare	Failure	# Rare	Failure	# Rare	Failure
20	3	33.3%	3	0%	4	25%
25	4	25%	4	0%	5	20%
30	12	0%	15	0%	18	16.7%
35	20	0%	24	0%	28	4.2%
40	36	0%	42	0%	47	10.6%
45	56	1.8%	64	3.1%	69	2.9%

Table 8: Summary information for six different sized tables where all cells have expected frequencies ≥ 5

	No Constraint					
	$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.1$	
Sample size	# Rare	Failure	# Rare	Failure	# Rare	Failure
20	18	27.8%	22	9.1%	25	16%
25	26	11.5%	34	11.8%	37	13.5%
30	35	11.4%	43	9.3%	45	6.7%
35	45	13.3%	51	7.8%	54	5.6%
40	55	14.6%	62	6.5%	66	6.1%
45	58	10.3%	65	7.7%	68	2.9%

Table 9: Summary information for six different sized tables where at least one cell has expected < 5

7 Interpreting the Results

7.1 Rule #1, when sample size ≤ 20

The biggest concern with a sample size smaller than 20 is that there are only a few tables, for any given sample, which would be classified as rare. Therefore, is is next to impossible to control one's type I error with any thing other than an exact test. This is very clear for the N=8 case where there are only three possible rare tables, all of which could be misclassified by at least one of the three decision rules based on asymptotic test.

Increasing sample size, in our case to 19, increases the number of true rare tests, but doesn't remove the presence of several misclassifications. However, no misclassifications were made when the ratio to the diagonal cell frequencies was greater than 0.25. So for a 2×2 table of size 19, if the ratio of the diagonal frequencies is greater than 0.25, a χ^2 test holds the type I error to any of the three previously mentioned decision rules. However, similar ratio rules were not found in any of the larger samples so one might assume that it likewise doesn't work for any samples smaller than 19.

Based on these two samples it is concluded that this rule of thumb generally is a good rule to follow. Therefore, it is recommended that for samples smaller than 20 only an exact procedure be used.

7.2 Rule #2, when $20 \leq N \leq 40$

As we increase the sample size, case failure rates do decrease. Implying that large sample theory is drawing the approximated p-value closer to the true p-value. Unfortunately, sample sizes found with in the range of this rule of thumb, still behave poorly when the approximated p-value is used in decision makeing.

For these samples the rule works for $\alpha = 0.05$, that is, the asymptotic type I error is held to some value which is less than or equal to 0.05. It is odd to note that for samples of size 45 the asymptotic method has a 3.1% failure. If we look at the No Constraint data we see that failure rates are all smaller for the constrained test.

The rule has mixed results for an $\alpha = 0.01$ test. For this decision rule the rule works completely for sample sizes of 30, 35, and 40. We haven't investigated what happens for intermediate sample sizes, say 32 or 39. However, an asymptotic test for samples of either 20 or 25 would be a grave mistake. For samples which don't meet the frequency constraint, asymptotic methods are also a poor choice. Even for a sample of size 45 there is a large failure rate for those tables which don't meet the constraint . For an $\alpha = 0.01$ test there is very little application for the asymptotic test, and an exact test would be a better test.

The last decision rule of $\alpha = 0.10$ fails to perform well for either the constrained or the non-constrained tables, and an exact p-value is highly recommended to assure that the type I error is contained at 0.10.

This rule of thumb has only limited uses, and although a rule of thumb is only a general idea to follow, it often tells people that there are no problems with test results if the rule is satisfied. Thus, this rule is only a good one if the decision rule is held at $\alpha = 0.05$. Any other decision rule should be used together with an exact p-value.

8 Conclusions and Recommendations

It has been successfully demonstrated that for small sample sizes tabled data must be analyzed with the exact procedure, whether done by hand and table, SAS, or a package like StatXact. Unfortunately, most tables are only tabulated up to sample sizes of 20 while some computer packages have sample size limited exact procedures. So when is the sample size large enough so that a χ^2 approximation doesn't give any false significant results? From the rules of thumb investigated it appears that anything larger than 40 should be free from any incorrect results. However, this report shows that even for a sample size of 45 the type I error rate isn't in the full control of the researcher. Therefore, I believe a useful area of study could be to determine if a specific sample size exists which would play as a cut-off-point for decision rule agreement in exact and approximated p-values. It might be that such a cutt-off doesn't exist. For examplt it is possible that for some large sample sample size the exact p-value be 0.0501 and the approximated p-value be 0.0499, one significant and another not significant at the $\alpha = 0.05$ level.

It is only reasonable to suggest that for results which might have a strong impact on society the researcher should validate their results with an exact procedure. Obviously, this implies that packages like StatXact will and must become more accessible. Additional study should be done to address the reliability and the limits of this package.

From this report I have been impressed by the discrepancies in the two p-value generating methods. I must conclude that I would be very hesitant to do any type of contingency analysis for a sample size smaller than 40 with any thing other than an exact procedure. What happens for higher ordered tables? On a few that I have looked at I have been intrigued to see that the p-values are switched. That is, the approximated p-value is more conservative than the exact p-value. Possible further investigation could be done to see the behavior of the 3×3 tables or others.

9 Appendix–References & Tables

References

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Sample size = 30					
Case 1: E[smallest cell] = 5.6			Case 2: E[smallest cell] = 6.5		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
17:13	0.0000	0.0000	16:14	0.0000	0.0000
16:12	0.0000	0.0000	15:13	0.0000	0.0000
15:11	0.0001	0.0001	14:12	0.0001	0.0001
14:10	0.0027	0.0011	13:11	0.0027	0.0011
13:9	0.0247	0.0123	12:10	0.0261	0.0110
12:8	0.1376	0.0785	11:9	0.1414	0.0704
Case 3: E[smallest cell] = 7.5					
Diagonals	Exact	Asymptotic			
15:15	0.0000	0.0000			
14:14	0.0000	0.0000			
13:13	0.0001	0.0001			
12:12	0.0028	0.0010			
11:11	0.0268	0.0106			
10:10	0.1431	0.0679			

Table 10: Tables of size 30 where $E[count] \geq 5$

Sample size = 35					
Case 1: E[smallest cell] = 5.6			Case 2: E[smallest cell] = 6.4		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
21:14	0.0000	0.0000	20:15	0.0000	0.0000
20:13	0.0000	0.0000	19:14	0.0000	0.0000
19:12	0.0000	0.0000	18:13	0.0000	0.0000
18:11	0.0003	0.0001	17:12	0.0002	0.0001
17:10	0.0041	0.0019	16:11	0.0024	0.0016
16:9	0.0332	0.0166	15:10	0.0192	0.0137
15:8	0.1587	0.0910	14:9	0.0966	0.0759
Case 3: E[smallest cell] = 7.3			Case 4: E[smallest cell] = 8.3		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
19:16	0.0000	0.0000	18:17	0.0000	0.0000
18:15	0.0000	0.0000	17:16	0.0000	0.0000
17:14	0.0000	0.0000	16:15	0.0000	0.0000
16:13	0.0002	0.0001	15:14	0.0002	0.0001
15:12	0.0022	0.0014	14:13	0.0022	0.0013
14:11	0.0185	0.0121	13:12	0.0184	0.0113
13:10	0.0946	0.0674	12:11	0.0943	0.0634

Table 11: Tables of size 35 where $E[count] \geq 5$

Sample size = 40					
Case 1: E[smallest cell] = 5.6			Case 2: E[smallest cell] = 6.4		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
25:15	0.0000	0.0000	24:16	0.0000	0.0000
24:14	0.0000	0.0000	23:15	0.0000	0.0000
23:13	0.0000	0.0000	22:14	0.0000	0.0000
22:12	0.0000	0.0000	21:13	0.0000	0.0000
21:11	0.0005	0.0003	20:12	0.0003	0.0002
20:10	0.0062	0.0032	19:11	0.0036	0.0024
19:9	0.0418	0.0228	18:10	0.0245	0.0177
18:8	0.1775	0.1091	17:9	0.1098	0.0867
Case 3: E[smallest cell] = 7.2			Case 4: E[smallest cell] = 8.1		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
23:17	0.0000	0.0000	22:18	0.0000	0.0000
22:16	0.0000	0.0000	21:17	0.0000	0.0000
21:15	0.0000	0.0000	20:16	0.0000	0.0000
20:14	0.0000	0.0000	19:15	0.0000	0.0000
19:13	0.0003	0.0002	18:14	0.0003	0.0002
18:12	0.0034	0.0020	17:13	0.0035	0.0017
17:11	0.0237	0.0146	16:12	0.0244	0.0127
16:10	0.1082	0.0726	15:11	0.1098	0.0639
Case 5: E[smallest cell] = 9.0			Case 6: E[smallest cell] = 10.0		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
21:19	0.0000	0.0000	20:20	0.0000	0.0000
20:18	0.0000	0.0000	19:19	0.0000	0.0000
19:17	0.0000	0.0000	18:18	0.0000	0.0000
18:16	0.0000	0.0000	17:17	0.0000	0.0000
17:15	0.0003	0.0002	16:16	0.0004	0.0001
16:14	0.0037	0.0016	15:15	0.0038	0.0016
15:13	0.0253	0.0117	14:14	0.0256	0.0114
14:12	0.1119	0.0593	13:13	0.1128	0.0578

Table 12: Tables of size 40 where $E[count] \geq 5$

Sample size = 45					
Case 1: E[smallest cell] = 5.0			Case 2: E[smallest cell] = 5.7		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
30:15	0.0000	0.0000	29:16	0.0000	0.0000
25:10	0.0018	0.0008	24:11	0.0010	0.0005
24:9	0.0167	0.0073	23:10	0.0088	0.0050
23:8	0.0912	0.0442	22:9	0.0508	0.0312
Case 3: E[smallest cell] = 6.4			Case 4: E[smallest cell] = 7.2		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
28:17	0.0000	0.0000	27:18	0.0000	0.0000
23:12	0.0006	0.0004	22:13	0.0005	0.0003
22:11	0.0053	0.0037	21:12	0.0049	0.0029
21:10	0.0308	0.0233	20:11	0.0296	0.0183
20:9	0.1021	0.1240	19:10	0.1220	0.0820
Case 5: E[smallest cell] = 8.0			Case 6: E[smallest cell] = 8.9		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
26:19	0.0000	0.0000	25:20	0.0000	0.0000
21:14	0.0006	0.0003	20:15	0.0003	0.0002
20:13	0.0052	0.0024	19:14	0.0029	0.0020
19:12	0.0311	0.0151	18:13	0.0177	0.0131
18:11	0.1256	0.0688	17:12	0.0767	0.0603
Case 7: E[smallest cell] = 9.8			Case 8: E[smallest cell] = 10.8		
Diagonals	Exact	Asymptotic	Diagonals	Exact	Asymptotic
24:21	0.0000	0.0000	23:22	0.0000	0.0000
19:16	0.0003	0.0002	18:17	0.0003	0.0002
18:15	0.0028	0.0018	17:16	0.0028	0.0018
17:14	0.0174	0.0119	16:15	0.0174	0.0113
16:13	0.0759	0.0553	15:14	0.0758	0.0529

Table 13: Tables of size 45 where $E[count] \geq 5$

Sample size = 20					
Case 1: E[smallest cell]=4.05			Case 2: E[smallest cell]=3.2		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
11:9	0.0000	0.0000	12:8	0.0000	0.0000
10:8	0.0009	0.0004	11:7	0.0008	0.0004
9:2	0.0216	0.0077	10:6	0.0194	0.0091
8:6	0.1748	0.0781	9:5	0.1675	0.0935
Case 3: E[smallest cell]=2.45			Case 4: E[smallest cell]=1.8		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
13:7	0.0000	0.0000	14:6	0.0000	0.0000
12:6	0.0012	0.0005	13:5	0.0022	0.0007
11:5	0.0223	0.0122	12:4	0.0374	0.0192
10:4	0.1736	0.1276	11:3	0.3027	0.2013
Case 5: E[smallest cell]=1.25			Case 6: E[smallest cell]=0.8		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
15:5	0.0000	0.0000	16:4	0.0000	0.0002
14:4	0.0049	0.0010	15:3	0.0134	0.0021
13:3	0.0726	0.0369	14:2	0.1620	0.0935
Case 7: E[smallest cell]=0.45			Case 8: E[smallest cell]=0.2		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
17:3	0.0009	0.0000	18:2	0.0053	0.0000
16:2	0.0456	0.0066	17:1	0.1947	0.0469
Case 9: E[smallest cell]=0.05					
Diag	Exact	Asymptotic			
19:1	0.0000	0.0500			

Table 14: Tables of size 20 where $E[count] < 5$

Sample size = 25					
Case 1: E[smallest cell]=4.84			Case 2: E[smallest cell]=4.0		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
14:11	0.0000	0.0000	15:10	0.0000	0.0000
13:10	0.0000	0.0000	14:9	0.0000	0.0000
12:9	0.0012	0.0007	13:8	0.0024	0.0009
11:8	0.0172	0.0103	12:7	0.0344	0.0124
10:7	0.1160	0.0796	11:6	0.2107	0.0956
Case 3: E[smallest cell]=3.2			Case 4: E[smallest cell]=2.6		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
16:9	0.0000	0.0000	17:8	0.0000	0.0000
15:8	0.0001	0.0000	16:7	0.0001	0.0000
14:7	0.0022	0.0011	15:6	0.0036	0.0016
13:6	0.0308	0.0166	14:5	0.0613	0.0249
12:5	0.1998	0.1266			
Case 5: E[smallest cell]=1.96			Case 6: E[smallest cell]=1.44		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
18:7	0.0000	0.0000	19:6	0.0000	0.0000
17:6	0.0003	0.0001	18:5	0.0006	0.0001
16:5	0.0069	0.0026	17:4	0.0151	0.0050
15:4	0.0664	0.0430	16:3	0.1246	0.0872
Case 7: E[smallest cell]=1.0			Case 8: E[smallest cell]=0.64		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
20:5	0.0000	0.0000	21:4	0.0000	0.0000
19:4	0.0019	0.0002	20:3	0.0067	0.0004
18:3	0.0377	0.0124	19:2	0.1064	0.0430
Case 9: E[smallest cell]=0.36			Case 10: E[smallest cell]=0.16		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
22:3	0.0004	0.0000	23:2	0.0033	0.0000
21:2	0.0291	0.0019	22:1	0.1567	0.0225
Case 11: E[smallest cell]=0.04					
Diag	Exact	Asymptotic			
24:1	0.0400	0.0000			

Table 15: Tables of size 25 where $E[count] < 5$

Sample size = 30					
Case 1: E[smallest cell]=4.8			Case 2: E[smallest cell]=4.03		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
18:12	0.0000	0.0000	19:11	0.0000	0.0000
17:11	0.0000	0.0000	18:10	0.0000	0.0000
16:10	0.0001	0.0001	17:9	0.0002	0.0001
15:9	0.0024	0.0014	16:8	0.0045	0.0018
14:8	0.0243	0.0149	15:7	0.0465	0.0197
13:7	0.1362	0.0942			
Case 3: E[smallest cell]=3.3			Case 4: E[smallest cell]=2.7		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
20:10	0.0000	0.0000	21:9	0.0000	0.0000
19:9	0.0000	0.0000	20:8	0.0000	0.0000
18:8	0.0003	0.0001	19:7	0.0005	0.0002
17:7	0.0048	0.0026	18:6	0.0084	0.0041
16:6	0.0449	0.0285	17:5	0.0816	0.0455
Case 5: E[smallest cell]=2.1			Case 6: E[smallest cell]=1.6		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
22:8	0.0000	0.0000	23:7	0.0000	0.0000
21:7	0.0000	0.0000	22:6	0.0001	0.0000
20:6	0.0011	0.0003	21:5	0.0027	0.0006
19:5	0.0159	0.0074	20:4	0.0331	0.0157
18:4	0.1580	0.0814			
Case 7: E[smallest cell]=1.2			Case 8: E[smallest cell]=0.8		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
24:6	0.0000	0.0000	25:5	0.0000	0.0000
23:5	0.0002	0.0000	24:3	0.0009	0.0000
22:4	0.0072	0.0014	23:2	0.0219	0.0044
21:3	0.0754	0.0400			
Case 9: E[smallest cell]=0.5			Case 10: E[smallest cell]=0.3		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
26:4	0.0000	0.0000	27:3	0.0002	0.0000
25:3	0.0038	0.0001	26:2	0.0202	0.0006
24:2	0.0750	0.0205			

Sample size = 30					
Case 11: E[smallest cell]=0.1			Case 12: E[smallest cell]=0.03		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
28:2	0.0023	0.0000	29:1	0.0333	0.0000
27:1	0.1310	0.0110			

Table 16: Tables of size 30 where $E[count] < 5$

Sample size = 35					
Case 1: E[smallest cell]=4.8			Case 2: E[smallest cell]=4.1		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
22:13	0.0000	0.0000	23:12	0.0000	0.0000
21:12	0.0000	0.0000	22:11	0.0000	0.0000
20:11	0.0000	0.0000	21:10	0.0000	0.0000
19:10	0.0003	0.0002	20:9	0.0005	0.0002
18:9	0.0042	0.0025	19:8	0.0074	0.0036
17:8	0.0328	0.0217	18:7	0.0587	0.0304
Case 3: E[smallest cell]=3.6			Case 4: E[smallest cell]=2.8		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
24:11	0.0000	0.0000	25:10	0.0000	0.0000
23:10	0.0000	0.0000	24:9	0.0000	0.0000
22:9	0.0000	0.0000	23:8	0.0001	0.0000
21:8	0.0008	0.0004	22:7	0.0016	0.0006
20:7	0.0092	0.0055	21:6	0.0160	0.0092
19:6	0.0623	0.0461	20:5	0.1068	0.0759
Case 5: E[smallest cell]=2.3			Case 6: E[smallest cell]=1.8		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
26:9	0.0000	0.0000	27:8	0.0000	0.0000
25:8	0.0000	0.0000	26:7	0.0000	0.0000
24:7	0.0002	0.0000	25:6	0.0004	0.0001
23:6	0.0033	0.0011	24:5	0.0074	0.0024
22:5	0.0299	0.0175	23:4	0.0596	0.0374
Case 7: E[smallest cell]=1.4			Case 8: E[smallest cell]=1.0		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
28:7	0.0000	0.0000	29:6	0.0000	0.0000
27:6	0.0000	0.0000	28:5	0.0001	0.0000
26:5	0.0012	0.0001	27:4	0.0039	0.0004
25:4	0.0183	0.0060	26:3	0.0489	0.0190
24:3	0.1248	0.0910			

Sample size = 35					
Case 9: E[smallest cell]=0.7			Case 10 E[smallest cell]=0.5		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
30:5	0.0000	0.0000	31:4	0.0000	0.0000
29:4	0.0005	0.0000	30:3	0.0024	0.0000
28:3	0.0139	0.0016	29:2	0.0557	0.0100
27:2	0.1389	0.0759			
Case 11: E[smallest cell]=0.3			Case 12 E[smallest cell]=0.1		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
32:3	0.0002	0.0000	33:2	0.0017	0.0000
31:2	0.0148	0.0002			
Case 13: E[smallest cell]=0.02					
Diag	Exact	Asymptotic			
34:1	0.0286	0.0000			

Table 17: Tables of size 35 where $E[count] < 5$

Sample size = 40					
Case 1: E[smallest cell]=4.9			Case 2: E[smallest cell]=4.23		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
26:14	0.0000	0.0000	27:13	0.0000	0.0000
25:13	0.0000	0.0000	26:12	0.0000	0.0000
24:12	0.0000	0.0000	25:11	0.0000	0.0000
23:11	0.0000	0.0000	24:10	0.0001	0.0000
22:10	0.0007	0.0004	23:9	0.0011	0.0006
21:9	0.0068	0.0044	22:8	0.0114	0.0065
20:8	0.0428	0.0312	21:7	0.0724	0.0455
Case 3: E[smallest cell]=3.6			Case 4: E[smallest cell]=3.03		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
28:12	0.0000	0.0000	29:11	0.0000	0.0000
27:11	0.0000	0.0000	28:10	0.0000	0.0000
26:10	0.0000	0.0000	27:9	0.0000	0.0000
25:9	0.0001	0.0000	26:8	0.0003	0.0000
24:8	0.0019	0.0009	25:7	0.0037	0.0016
23:7	0.0213	0.0105	24:6	0.0424	0.0183
22:6	0.1298	0.0708			
Case 5: E[smallest cell]=2.5			Case 6: E[smallest cell]=2.03		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
30:10	0.0000	0.0000	31:9	0.0000	0.0000
29:9	0.0000	0.0000	30:8	0.0000	0.0000
28:8	0.0000	0.0000	29:7	0.0001	0.0000
27:7	0.0006	0.0001	28:6	0.0014	0.0003
26:6	0.0074	0.0032	27:5	0.0159	0.0070
25:5	0.0852	0.0350	26:4	0.1680	0.0733
Case 7: E[smallest cell]=1.6			Case 8: E[smallest cell]=1.23		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
32:8	0.0000	0.0000	33:7	0.0000	0.0000
31:7	0.0000	0.0000	32:6	0.0000	0.0000
30:6	0.0002	0.0000	31:5	0.0006	0.0000
29:5	0.0038	0.0008	30:4	0.0108	0.0024
28:4	0.0365	0.0177	29:3	0.0877	0.0579

Sample size = 40					
Case 9: E[smallest cell]=0.9			Case 10: E=0.63		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
34:6	0.0000	0.0000	35:5	0.0000	0.0000
33:5	0.0001	0.0000	34:4	0.0003	0.0000
32:4	0.0022	0.0001	33:3	0.0093	0.0006
31:3	0.0334	0.0092	32:2	0.1088	0.0468
Case 11: E[smallest cell]=0.4			Case 12: E[smallest cell]=0.23		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
36:4	0.0000	0.0000	37:3	0.0001	0.0000
35:3	0.0016	0.0000	36:2	0.0113	0.0001
34:2	0.0429	0.0049	35:1	0.2136	0.0773
Case 13: E[smallest cell]=0.1			Case 14: E[smallest cell]=0.33		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
38:2	0.0013	0.0000	39:1	0.0250	0.0000
37:1	0.0987	0.0027			

Table 18: Tables of size 40 where $E[count] < 5$

Sample size = 45					
Case 1: E[smallest cell]=4.36			Case 2: E[smallest cell]=3.76		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
31:14	0.0000	0.0000	32:13	0.0000	0.0000
30:13	0.0000	0.0000	31:12	0.0000	0.0000
29:12	0.0000	0.0000	30:11	0.0000	0.0000
28:11	0.0000	0.0000	29:10	0.0000	0.0000
27:10	0.0002	0.0001	28:9	0.0004	0.0001
26:9	0.0027	0.0012	27:8	0.0039	0.0021
25:8	0.0171	0.0112	26:7	0.0300	0.0189
24:7	0.0885	0.0659			
Case 3: E[smallest cell]=3.2			Case 4: E[smallest cell]=2.7		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
33:12	0.0000	0.0000	34:11	0.0000	0.0000
32:11	0.0000	0.0000	33:10	0.0000	0.0000
31:10	0.0000	0.0000	32:9	0.0000	0.0000
30:9	0.0000	0.0000	31:8	0.0001	0.0000
29:8	0.0007	0.0003	30:7	0.0016	0.0005
28:7	0.0073	0.0038	29:6	0.0143	0.0075
27:6	0.0552	0.0328	28:5	0.1037	0.0621
Case 5: E[smallest cell]=2.2			Case 6: E[smallest cell]=1.8		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
35:10	0.0000	0.0000	36:9	0.0000	0.0000
34:9	0.0000	0.0000	35:8	0.0000	0.0000
33:8	0.0000	0.0000	34:7	0.0000	0.0000
32:7	0.0003	0.0000	33:6	0.0007	0.0001
31:6	0.0037	0.0011	32:5	0.0091	0.0029
30:5	0.0293	0.0166	31:4	0.0622	0.0404
Case 7: E[smallest cell]=1.4			Case 8: E[smallest cell]=1.1		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
37:8	0.0000	0.0000	38:7	0.0000	0.0000
36:7	0.0000	0.0000	37:6	0.0000	0.0000
35:6	0.0001	0.0000	36:5	0.0003	0.0000
34:5	0.0021	0.0003	35:4	0.0068	0.0010
33:4	0.0236	0.0086	34:3	0.0638	0.0301

Sample size = 45					
Case 9: E[smallest cell]=0.8			Case 10: E[smallest cell]=0.56		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
39:6	0.0000	0.0000	40:5	0.0000	0.0000
38:5	0.0000	0.0000	39:4	0.0002	0.0000
37:4	0.0014	0.0000	38:3	0.0065	0.0002
36:3	0.0238	0.0045	37:2	0.0874	0.0292
Case 11: E[smallest cell]=0.36			Case 12: E[smallest cell]=0.2		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
41:4	0.0000	0.0000	42:3	0.0001	0.0000
40:3	0.0011	0.0000	41:2	0.0089	0.0000
39:2	0.0341	0.0025	40:1	0.1910	0.0553
Case 13: E[smallest cell]=0.09			Case 14: E[smallest cell]=0.02		
Diag	Exact	Asymptotic	Diag	Exact	Asymptotic
43:2	0.0010	0.0000	44:1	0.0222	0.0000
42:1	0.0879	0.0014			

Table 19: Tables of size 45 where $E[count] < 5$